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PROBLEMS AND SOLUTIONS
IN
EUCLIDEAN GEOMETRY
PROBLEMS AND SOLUTIONS IN EUCLIDEAN GEOMETRY

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DOVER PUBLICATIONS, INC
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Dedicated to my father
who has been always of lifelong inspiration
and encouragement to me

M.N.A.
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PREFACE

This book is intended as a second course in Euclidean geometry. Its purpose is to give the reader facility in applying the theorems of Euclid to the solution of geometrical problems. Each chapter begins with a brief account of Euclid’s theorems and corollaries for simplicity of reference, then states and proves a number of important propositions. Chapters close with a section of miscellaneous problems of increasing complexity, selected from an immense mass of material for their usefulness and interest. Hints of solutions for a large number of problems are also included.

Since this is not intended for the beginner in geometry, such familiar concepts as point, line, ray, segment, angle, and polygon are used freely without explicit definition. For the purpose of clarity rather than rigor the general term line is used to designate sometimes a ray, sometimes a segment, sometimes the length of a segment; the meaning intended will be clear from the context.

Definitions of less familiar geometrical concepts, such as those of modern and space geometry, are added for clarity, and since the use of symbols might prove an additional difficulty to some readers, geometrical notation is introduced gradually in each chapter.

January, 1968

M. N. AREF

WILLIAM WERNICK
SYMBOLS

EMPLOYED IN THIS BOOK

\[ \angle \] angle
\[ \circ \] circle
\[ \square \] parallelogram
\[ \triangle \] quadrilateral
\[ \square \] rectangle
\[ \square \] square
\[ \triangle \] triangle

\[ a = b \] \( a \) equals \( b \)
\[ a > b \] \( a \) is greater than \( b \)
\[ a < b \] \( a \) is less than \( b \)
\[ a \parallel b \] \( a \) is parallel to \( b \)
\[ a \perp b \] \( a \) is perpendicular to \( b \)
\[ a / b \] \( a \) divided by \( b \)
\[ a : b \] the ratio of \( a \) to \( b \)
\[ AB^2 \] the square of the distance from \( A \) to \( B \)
\[ \therefore \] because
\[ \therefore \] therefore
CHAPTER 1

TRIANGLES AND POLYGONS

Theorems and Corollaries

LINES AND ANGLES

1.1. If a straight line meets another straight line, the sum of the two adjacent angles is two right angles.

**Corollary 1.** If any number of straight lines are drawn from a given point, the sum of the consecutive angles so formed is four right angles.

**Corollary 2.** If a straight line meets another straight line, the bisectors of the two adjacent angles are at right angles to one another.

1.2. If the sum of two adjacent angles is two right angles, their non-coincident arms are in the same straight line.

1.3. If two straight lines intersect, the vertically opposite angles are equal.

1.4. If a straight line cuts two other straight lines so as to make the alternate angles equal, the two straight lines are parallel.

1.5. If a straight line cuts two other straight lines so as to make: (i) two corresponding angles equal; or (ii) the interior angles, on the same side of the line, supplementary, the two straight lines are parallel.

1.6. If a straight line intersects two parallel straight lines, it makes: (i) alternate angles equal; (ii) corresponding angles equal; (iii) two interior angles on the same side of the line supplementary.

**Corollary.** Two angles whose respective arms are either parallel or perpendicular to one another are either equal or supplementary.

1.7. Straight lines which are parallel to the same straight line are parallel to one another.

TRIANGLES AND THEIR CONGRUENCE

1.8. If one side of a triangle is produced, (i) the exterior angle is equal to the sum of the interior non-adjacent angles; (ii) the sum of the three angles of a triangle is two right angles.

**Corollary 1.** If two angles of one triangle are respectively equal to two angles of another triangle, the third angles are equal and the triangles are equiangular.

**Corollary 2.** If one side of a triangle is produced, the exterior angle is greater than either of the interior non-adjacent angles.
Corollary 3. The sum of any two angles of a triangle is less than two right angles.

1.9. If all the sides of a polygon of $n$ sides are produced in order, the sum of the exterior angles is four right angles.

Corollary. The sum of all the interior angles of a polygon of $n$ sides is $(2n - 4)$ right angles.

1.10. Two triangles are congruent if two sides and the included angle of one triangle are respectively equal to two sides and the included angle of the other.

1.11. Two triangles are congruent if two angles and a side of one triangle are respectively equal to two angles and the corresponding side of the other.

1.12. If two sides of a triangle are equal, the angles opposite to these sides are equal.

Corollary 1. The bisector of the vertex angle of an isosceles triangle, (i) bisects the base; (ii) is perpendicular to the base.

Corollary 2. An equilateral triangle is also equiangular.

1.13. If two angles of a triangle are equal, the sides which subtend these angles are equal.

Corollary. An equiangular triangle is also equilateral.

1.14. Two triangles are congruent if the three sides of one triangle are respectively equal to the three sides of the other.

1.15. Two right-angled triangles are congruent if the hypotenuse and a side of one are respectively equal to the hypotenuse and a side of the other.

Inequalities

1.16. If two sides of a triangle are unequal, the greater side has the greater angle opposite to it.

1.17. If two angles of a triangle are unequal, the greater angle has the greater side opposite to it.

1.18. Any two sides of a triangle are together greater than the third.

1.19. If two triangles have two sides of the one respectively equal to two sides of the other and the included angles unequal, then the third side of that with the greater angle is greater than the third side of the other.

1.20. If two triangles have two sides of the one respectively equal to two sides of the other, and the third sides unequal, then the angle contained by the sides of that with the greater base is greater than the corresponding angle of the other.

1.21. Of all straight lines that can be drawn to a given straight line from a given external point, (i) the perpendicular is least; (ii) straight lines which make equal angles with the perpendicular are equal; (iii) one making a greater angle with the perpendicular is greater than one making a lesser angle.
Corollary. Two and only two straight lines can be drawn to a given straight line from a given external point, which are equal to one another.

Quadrilaterals and Over Four-sided Polygons

1.22. The opposite sides and angles of a parallelogram are equal, each diagonal bisects the parallelogram, and the diagonals bisect one another.

Corollary 1. The distance between a pair of parallel straight lines is everywhere the same.

Corollary 2. The diagonals of a rhombus bisect each other at right angles.

Corollary 3. A square is equilateral.

1.23. A quadrilateral is a parallelogram if (i) one pair of opposite sides are equal and parallel; (ii) both pairs of opposite sides are equal or parallel; (iii) both pairs of opposite angles are equal; (iv) the diagonals bisect one another.

1.24. Two parallelograms are congruent if two sides and the included angle of one are equal respectively to two sides and the included angle of the other.

Corollary. Two rectangles having equal bases and equal altitudes are congruent.

1.25. If three or more parallel straight lines intercept equal segments on one transversal, they intercept equal segments on every transversal.

Corollary. A line parallel to a base of a trapezoid and bisecting a leg bisects the other leg also.

1.26. If a line is drawn from the mid-point of one side of a triangle parallel to the second side, it bisects the third side. This line is called a mid-line of a triangle.

Corollary 1. Conversely, a mid-line of a triangle is parallel to the third side and is equal to half its magnitude.

Corollary 2. In any triangle, a mid-line between two sides and the median to the third side bisect each other.

1.27. In a right triangle, the median from the right vertex to the hypotenuse is equal to half the hypotenuse.

1.28. If one angle of a right triangle is 30°, the side opposite this angle is equal to half the hypotenuse.

Corollary. Conversely, if one side of a right triangle is half the hypotenuse, the angle opposite to it is 30°.

1.29. The median of a trapezoid is parallel to the parallel bases and is equal to half their sum.

Corollary. The line joining the mid-points of the diagonals of a trapezoid is parallel to the parallel bases and is equal to half their difference.
1.30. In an isosceles trapezoid, the base angles and the diagonals are equal to one another.

**Introduction to Concurrency**

1.31. The perpendicular bisectors of the sides of a triangle are concurrent in a point equidistant from the vertices of the triangle which is the center of the circumscribed circle and called the circumcenter of the triangle.

1.32. The bisectors of the angles of a triangle are concurrent in a point equidistant from the sides of the triangle which is the center of the inscribed circle and called the incenter of the triangle.

**Corollary 1.** The bisector of any interior angle and the external bisectors of the other, two exterior angles are concurrent in a point outside the triangle which is equidistant from the sides (or produced) of the triangle and called an excenter of the triangle.

**Corollary 2.** There are four points equidistant from the three sides of a triangle: one inside the triangle, which is the incenter, and three outside it, which are the excenters.

1.33. The altitudes of a triangle are concurrent in a point called the orthocenter of the triangle.

1.34. The medians of a triangle are concurrent in a point \( \frac{2}{3} \) the distance from each vertex to the mid-point of the opposite side. This point is called the centroid of the triangle.

**Solved Problems**

1.1. \( ABC \) is a triangle having \( BC = 2 \ AB \). Bisect \( BC \) in \( D \) and \( BD \) in \( E \). Prove that \( AD \) bisects \( \angle CAE \).

**Construction:** Draw \( DF \parallel AC \) to meet \( AB \) in \( F \) (Fig. 1.)

**Proof:** :: \( D \) is the mid-point of \( BC \) and \( DF \parallel AC \), :: \( F \) is the mid-point of \( AB \) (Th. 1.26). Also, \( AB = BD = \frac{1}{2} BC \). :: \( BF = BE \). :: \( \triangle ABE \) and \( DBF \) are congruent. :: \( \angle EAF = \angle EDF \), but \( \angle BAD = \angle BDA \) (since \( BA = BD \), Th. 1.12). :: Subtraction gives \( \angle EAD \)
1.2. ABC is a triangle. D and E are any two points on AB and AC. The bisectors of the angles ABE and ACD meet in F. Show that \( \angle BDC + \angle BEC = 2 \angle BFC \).

**Construction:** Join AF and produce it to meet BC in G (Fig. 2).

**Proof:** \( \angle BDC \) is exterior to \( \triangle ADC \). \( \therefore \angle BDC = \angle A + \angle ACD \) (Th. 1.8). Also, \( \angle BEC \) is exterior to \( \triangle AEB \). \( \therefore \angle BEC = \angle A + \angle ABE; \) hence \( \angle BDC + \angle BEC = 2 \angle A + \angle ACD + \angle ABE \) (1).

Similarly, \( \angle BFG, CFG \) are exterior to \( \triangle AFB, AFC \). \( \therefore \angle BFG + \angle CFG = \angle BFC = \angle A + \frac{1}{2} \angle ABE + \frac{1}{2} \angle ACD \) (2). Therefore, from (1) and (2), \( \angle BDC + \angle BEC = 2 \angle BFC \).

1.3. ABC is a right-angled triangle at A and \( AB > AC \). Bisect BC in D draw DE perpendicular to the hypotenuse BC to meet the bisector of the right angle A in E. Prove that (i) \( AD = DE \); (ii) \( \angle DAE = \frac{1}{2} (\angle C - \angle B) \).

**Construction:** Draw \( AF \perp BC \) (Fig. 3).
Proof: (i) \( \because \) D is the mid-point of BC. \( \therefore \) \( AD = DB = DC \) (Th. 1.27). \( \therefore \) \( \angle DAB = \angle DBA \). But, \( \angle DBA \) or \( FBA = \angle FAC \) (since they are complementary to \( \angle FAB \)). \( \therefore \) \( \angle DAB = \angle FAC \). But since \( AO \) bisects \( \angle A \), \( \therefore \) \( \angle OAB = \angle OAC \) and subtraction gives \( \angle OAD = \angle OAF \). \( \therefore \) \( DE \parallel AF \) (both \( \perp BC \)), \( \therefore \) \( \angle OAF = \angle OED \), and hence \( \angle OAD = \angle OED \). \( \therefore \) \( AD = DE \) (Th. 1.13).

(ii) As shown above, \( \angle DAB = \angle DBA \) or \( B \). Since \( \angle FAB = \angle C \) (both complement \( \angle FAC \)), \( \therefore \) \( (\angle FAB - \angle DAB) = (\angle C - \angle B) = 2\angle DAO \) or \( \angle DAE = \frac{1}{2} (\angle C - \angle B) \).

1.4. The sides \( AB \), \( BC \), and \( AC \) of a triangle are bisected in \( F \), \( G \) and \( H \) respectively. If \( BE \) is drawn perpendicular to \( AC \), prove that \( \angle FEG = \angle FHG \) (Fig. 4).

![Figure 4](image)

Proof: \( \because \) \( G \) and \( H \) are the mid-points of \( BC \) and \( AC \). \( \therefore \) \( HG \parallel AB \) and = its half (Cor. 1, Th. 1.26), or \( HG \parallel FB \) and equal to it. Hence \( FBGH \) is a \( \square \). \( \therefore \) \( \angle FHG = \angle B \). But \( \triangle AEB \) is right angled at \( E \) and \( F \) is the mid-point of \( AB \). \( \therefore \) \( EF = FB \). \( \therefore \) \( \angle FEB = \angle FBE \) and, similarly, \( \angle GEB = \angle GBE \). Therefore, by adding, \( \angle FEG = \angle B = \angle FHG \).

1.5. In an isosceles triangle \( ABC \), the vertex angle \( A \) is \( \frac{\pi}{2} \) right angle. Let \( D \) be any point on the base \( BC \) and take a point \( E \) on \( CA \) so that \( CE = CD \). Join \( ED \) and produce it to meet \( AB \) produced in \( F \) (Fig. 5). If the bisector of the \( \angle EDC \) meets \( AC \) in \( R \), show that (i) \( AE = EF \); (ii) \( DR = DC \).

Proof: (i) \( \angle A = \frac{\pi}{2} \) right angle. \( \therefore \) The sum of the 3 \( \angle s \) of \( \triangle ABC \) = 2 right angles (Th. 1.8), half the difference is \( \angle B = \angle C = \frac{\pi}{2} \) right angle. But, in \( \triangle CED \), \( CE = CD \). \( \therefore \) \( \angle CED = \angle CDE = \frac{1}{2} (2 - \frac{\pi}{2}) \) right angle = \( \frac{\pi}{4} \) right angle. \( \therefore \) \( \angle CED \) is exterior to the \( \triangle AEF \), \( \therefore \) \( \angle CED = \angle A + \angle AFE \). But \( \angle A = \frac{\pi}{2} \) right angle. \( \therefore \) \( \angle AFE = \frac{\pi}{2} \) right angle or \( \angle A = \angle AFE \). \( \therefore \) \( AE = EF \) (Th. 1.13).

(ii) \( \therefore \) \( \angle CDE = \frac{\pi}{4} \) right angle, \( \therefore \) \( \angle CDR = \frac{\pi}{4} \) right angle. But \( \angle C = \frac{\pi}{2} \) right angle, hence in the \( \triangle CDR \), the difference from 2 right
angles would be $\angle CRD = \frac{\pi}{2}$ right angle. $\therefore \angle CRD = \angle C$. $\therefore DR = DC$ (Th. 1.13).

**Figure 5**

1.6. The vertex $C$ is a right angle in the triangle $ABC$. If the points $D$ and $E$ are taken on the hypotenuse, so that $BC = BD$ and $AC = AE$, show that $DE$ equals the sum of the perpendiculars from $D$ and $E$ on $AC$ and $BC$ respectively.

**Construction:** Draw $CN \perp AB$ and join $CD$ and $CE$ (Fig. 6).

**Figure 6**

**Proof:** $\because BC = BD$. $\therefore \angle BCD = \angle BDC$ (Th. 1.12). But $\angle BCD = \angle BCN + \angle NCD$ and $\angle BDC = \angle DCA + \angle A$ (Th. 1.8). Hence $\angle BCN + \angle NCD = \angle DCA + \angle A$ (1) and $\therefore CN \perp AB$, $\therefore \angle BCN = \angle A$ (since both complement $\angle NCA$) (2). From (1) and (2), $\therefore \angle NCD = \angle DCA$. But $\angle CND = \angle CFD = \text{right angle}$. $\therefore \triangle CND, CFD$ are congruent. $\therefore DN = DF$. Similarly, $EN = EG$ and, by adding, $\therefore DE = DF + EG$.

1.7. The side $AB$ in the rectangle $ABCD$ is twice the side $BC$. A point $P$ is taken on the side $AB$ so that $BP = \frac{1}{4} AB$. Show that $BD$ is perpendicular to $CP$. 
Construction: Bisect $AB$ in $Q$ and draw $QR \perp AB$ to meet $BD$ in $R$ (Fig. 7).

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure7.png}
\caption{Figure 7}
\end{figure}

Proof: $AB = 2 BC$, \therefore $BC = BQ$. In the $\triangle ABD$, $Q$ is the mid-point of $AB$ and $QR \parallel AD$. \therefore $R$ is the mid-point of $BD$ and $QR = \frac{1}{2} AD$ (Th. 1.26, Cor. 1). Since $BP = \frac{1}{2} AB = \frac{1}{2} AD$, \therefore $BP = QR$. Therefore, $\triangle BQR$ and $\triangle CBP$ are congruent. \therefore $\angle QBR = \angle BCP$. But since $\angle QBR + \angle RBC = \text{right angle}$, \therefore $\angle BCP + \angle RBC = \text{right angle}$, yielding $\angle BFC = \text{right angle}$ [Th. 1.8(ii)]. Hence $BD \perp CP$.

1.8. $ABC$ is a triangle. $BF$, $CG$ are any two lines drawn from the extremities of the base $BC$ to meet $AC$ and $AB$ in $F$ and $G$ respectively and intersect in $H$. Show that $AF + AG > HF + HG$.

Construction: Draw $HD$ and $HE \parallel$ to $AB$ and $AC$ respectively (Fig. 8).

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure8.png}
\caption{Figure 8}
\end{figure}

Proof: The figure $ADHE$ is a parallelogram [Th. 1.23(ii)]. \therefore $AD = HE$ and $AE = DH$. In the $\triangle DHF$, $DH + DF > HF$ (Th. 1.18).
\therefore $AE + DF > HF$ (1). Similarly in $\triangle EGH$, $EH + EG > HG$ or $AD + EG > HG$ (2). Adding (1) and (2) gives $AE + EG + AD + DF > HF + HG$. \therefore $AF + AG > HF + HG$.

1.9. $ABCD$ is a square. The bisector of the $\angle DBA$ meets the diagonal $AC$ in $F$. If $CK$ is drawn perpendicular to $BF$, intersecting $BD$ in $L$ and
produced to meet $AB$ in $R$, prove that $AR = 2SL$, where $S$ is the intersection of the diagonals.

**Construction:** From $S$ draw $ST \parallel CR$ to meet $BF$ in $J$ and $AB$ in $T$ (Fig. 9).

**Proof:** \( CR \perp BF, \therefore ST \perp BF \). $\triangle BLK$, $BK$ are congruent (Th. 1.11). \( \therefore BR = BL \). Again $\triangle BTJ$, $BSJ$ are congruent. \( \therefore BT = BS \). Hence $TR = SL$ and, since the two diagonals of a square bisect one another (Th. 1.22), \( S \) is the mid-point of $AC$. But $ST$ is $\parallel CR$ in the $\triangle ACR$. \( \therefore T \) is the mid-point of $AR$ (Th. 1.26). \( \therefore AR = 2TR \). \( \therefore AR = 2SL \).

1.10. The side $AB$ in the triangle $ABC$ is greater than $AC$, and $D$ is the mid-point of $BC$. From $C$ draw two perpendiculars to the bisectors of the internal and external vertical angles at $A$ to meet them in $F$ and $G$ respectively. Prove that $i)$ $DF = \frac{1}{2} (AB - AC)$; $ii)$ $DG = \frac{1}{2} (AB + AC)$.

**Construction:** Produce $CF$, $CG$ to meet $AB$, $BA$ produced in $H$ and $M$ (Fig. 10).
PROBLEMS AND SOLUTIONS IN EUCLIDEAN GEOMETRY

Proof: (i) \mile CF \perp AF and AF bisects the \angle A, then the \triangle s AFC, AFH are congruent. \mile CF = FH and AC = AH. In the \triangle CBH, D, F are the mid-points of BC and CH. \mile DF = \frac{1}{2} BH = \frac{1}{2} (AB - AH) = \frac{1}{2} (AB - AC).

(ii) Similarly, G is the mid-point of CM, and AC = AM. In the \triangle CBM, D, G are the mid-points of CB and CM. \mile DG = \frac{1}{2} BM = \frac{1}{2} (AB + AM) = \frac{1}{2} (AB + AC).

1.11. \triangle ABC is a triangle and \triangle AD is any line drawn from A to the base BC. From B and C, two perpendiculars BE and CF are drawn to AD or AD produced. If R is the mid-point of BC, prove that RE = RF.

Construction: Produce CF and BE so that CF = FG and BE = EH, then join AG, BG, AH, and CH (Fig. 11).

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure11.png}
\caption{Figure 11}
\end{figure}

Proof: \mile BE \perp AE and BE = EH (construct). Then \triangle s ABE, AHE are congruent (Th. 1.10). \mile AB = AH and \angle BAE = \angle HAE. Similarly, from the congruent \triangle s ACF, AGF: AC = AG and \angle CAF = \angle GAF. Therefore by subtracting, \angle BAG = \angle CAH. Hence \triangle s AGB, ACH are congruent. \mile BG = CH. In the \triangle CBG, R, F are the mid-points of CB and CG. \mile RF = \frac{1}{2} BG. Similarly, in the \triangle CBH, RE = \frac{1}{2} CH. Since BG = CH, \mile RE = RF.

1.12. Any finite displacement of a segment AB can be considered as though a rotation about a point called the pole. If the length of the segment and the angle which it rotates are given, describe the method to determine the pole. What would be the conditions of the angles between the rays around the pole?

Analysis: Let A_1B_1, A_2B_2 be the two positions of the given segment which is rotated a given angle (Fig. 12). Construct the two perpendicular bisectors CP_{12}, DP_{12} to A_1A_2, B_1B_2 respectively to
TRIANGLES AND POLYGONS

intersect at the required pole $P_{12}$. Since $\triangle s A_1 CP_{12}, A_2 CP_{12}$ are congruent (Th. 1.10), $P_{12}A_1 = P_{12}A_2$ and $\angle A_1 P_{12} C = \angle A_2 P_{12} C = \frac{1}{2}$ given $\angle A_1 P_{12} A_2$. Similarly, $P_{12}B_1 = P_{12}B_2$ and $\angle B_1 P_{12} D = \angle B_2 P_{12} D = \frac{1}{2}$ given $\angle B_1 P_{12} B_2$, but $\angle A_1 P_{12} A_2 = \angle B_1 P_{12} B_2$ = given. $\therefore \angle A_1 P_{12} B_1 = \angle A_2 P_{12} B_2$. Hence $\triangle s A_1 P_{12} B_1, A_2 P_{12} B_2$

are congruent; i.e., $A_1 B_1$ is rotated around the pole $P_{12}$ to a new position $A_2 B_2$ through the given $\angle s A_1 P_{12} A_2$ or $B_1 P_{12} B_2$.

SYNTHESIS: Join $A_1 A_2, B_1 B_2$ and construct their $\perp$ bisectors $C P_{12}, D P_{12}$ to meet at the required pole $P_{12}$. Notice that $\angle A_1 P_{12} B_1 = \angle A_2 P_{12} B_2 = \angle C P_{12} D$. This is called Chasle's theorem.

1.13. $A B C$ is a triangle in which the vertical angle $C$ is $60^\circ$. If the bisectors of the base angles $A, B$ meet $B C, A C$ in $P, Q$ respectively, show that $A B = A Q + B P$.

CONSTRUCTION: Take point $R$ on $A B$ so that $A R = A Q$. Join $R P$ and $R Q$. Let $A P, B Q$ intersect in $E$. Join $E R$ and $P Q$. Let $Q R$ intersect $A E$ in $D$ (Fig. 13).

Proof: Since $\angle C = 60^\circ$, $\therefore \angle A + \angle B = 120^\circ$ (Th. 1.8). Hence $\angle E A B + \angle E B A = 60^\circ$. $\therefore \angle A E B = 120^\circ$. $\therefore \angle P E B = \angle Q E A = 60^\circ$. Since $A R = A Q$ (construct) and $A D$ bisects $\angle A$, $\therefore \triangle s A D Q, A D R$ are congruent (Th. 1.10). $\therefore D Q = D R$ and $A D \perp Q R$. Hence $Q E = E R$ and $\angle Q E D = \angle R E D = 60^\circ$. $\therefore \angle R E B = \angle P E B = 60^\circ$. $\therefore E B$ bisects $\angle B$. $\therefore \triangle s E B P, E B R$ are congruent. $\therefore R B = B P$. 

FIGURE 12
1.14. In any quadrilateral the two lines joining the mid-points of each pair of opposite sides meet at one point, with the line joining the mid-points of its diagonals.

**Construction:** Let $ABCD$ be the quadrilateral and $E, F, G, H, K,$ and $L$ the mid-points of its sides and diagonals. Join $EF, FG, GH, HE, EK, KG, GL, LE, FK, KH, HL,$ and $LF$ (Fig. 14).

**Proof:** In the $\triangle ABC$, $E$ and $F$ are the mid-points of $AB$ and $BC$. $\therefore EF \parallel AC$ and equals its half (Th, 1.26. Cor. 1). Also, in the $\triangle DAC$, $HG \parallel AC$ and equals its half. Hence $EF = \text{and} \parallel HG$. $\therefore EFGH$ is a parallelogram (Th. 1.23). Therefore, its diagonals $EG$ and $FH$ bisect one another (Th. 1.22). Similarly, $FKHL$ and $EKGL$ are parallelograms also. $\therefore FH$ and $LK$ in the first $\square$ bisect one another, and $EG$ and $LK$ in the second $\square$ bisect one another also. Since the point of
intersection of $EG$, $FH$, and $LK$ is their mid-point, which cannot be more than one point, then $EG$ and $FH$ intersect at one point with the line $LK$.

1.15. $ABC$ is a triangle. If the bisectors of the two exterior angles $B$ and $C$ of the triangle meet at $D$ and $DE$ is the perpendicular from $D$ on $AB$ produced, prove that $AE = \frac{1}{2}$ the perimeter of the $\triangle ABC$.

**Construction:** Draw $DF$ and $DG \perp BC$ and $AC$ produced; then join $DA$ (Fig. 15).

**Figure 15**

**Proof:** $\triangle DBE$ and $DBF$ are congruent (Th. 1.11). $\therefore BE = BF$ and $DE = DF$. Similarly, $\triangle DCF$ and $DCG$ are congruent also. $\therefore CF = CG$ and $DG = DF$. Hence $DE = DG$. Therefore, $\triangle ADE$ and $ADG$ are congruent (Th. 1.15). $\therefore AE = AG$. But $AE = AB + BE = AB + BF$ and $AG = AC + CF$. $\therefore AE = \frac{1}{2} (AE + AG) = \frac{1}{2} (AB + BF + AC + CF) = \frac{1}{2} (AB + BC + AC)$. 

1.16. $AB$ is a straight line. $D$ and $E$ are any two points on the same side of $AB$. Find a point $F$ on $AB$ so that (i) the sum of $FD$ and $FE$ is a minimum; (ii) the difference between $FD$ and $FE$ is a minimum. When is this impossible?

**Construction:** (i) Draw $EN \perp AB$; produce $EN$ to $M$ so that $EN = NM$. Join $MD$ to cut $AB$ in the required point $F$ (Fig. 16).

**Proof:** Take another point $F'$ on $AB$ and join $EF$, $EF'$, $FD$, and $F'M$. $\therefore \triangle ENF$ and $MNF$ are congruent (Th. 1.10). $\therefore EF = FM$. $\therefore MD = FD + FE$. $\therefore D$ and $M$ are two fixed points, the line joining them is a minimum. This is clear from $F'$, since $EF' = F'M$ (from the congruence of $\triangle EF'N$ and $MF'N$) but $MF' + F'D > MD$ in the $\triangle DF'M$ (Th. 1.18). $\therefore EF' + F'D > EF + FD$. Hence $F$ is the required point and $(FD + FE)$ is a minimum.
CONSTRUCTION: (ii) Join $ED$. Bisect $ED$ in $N$, then draw $NF \perp ED$ to meet $AB$ in the required point $F$. Join $FD$ and $FE$ (Fig. 17).

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure16}
\caption{Figure 16}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure17}
\caption{Figure 17}
\end{figure}

Proof: $\triangle FDN$ and $FEN$ are congruent (Th. 1.10). $\therefore FD = FE$. Hence their difference will be zero and it is a minimum. This case will be impossible to solve when $ED$ is $\perp AB$, where $NF$ will be $\parallel AB$ and therefore $F$ cannot be determined.

1.17. Any straight line is drawn from $A$ in the parallelogram $ABCD$. $BE$, $CF$, $DG$ are $\perp$s from the other vertices to this line. Show that if the line lies outside the parallelogram, then $CF = BE + DG$, and when the line cuts the parallelogram, then $CF = \text{the difference between } BE \text{ and } DG$.

CONSTRUCTION: Join the diagonals $AC$ and $BD$ which intersect in $L$. Draw $LM \perp$ to the line through $A$ (Fig. 18).

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure18}
\caption{Figure 18}
\end{figure}
Proof: (i) The line lies outside the $\square ABCD$. The diagonals bisect one another (Th. 1.22). $\therefore L$ is the mid-point of $AC$ and $BD$. In the trapezoid $BDGE$, $L$ is the mid-point of $BD$ and $LM$ is $\parallel$ to $BE$ and $DG$ ($\perp$ s to the line $AG$). Hence $M$ is the mid-point of $GE$ and $LM = \frac{1}{2} (BE + DG)$ (Th. 1.29). Again in the $\triangle ACF$, $L$ is the mid-point of $AC$ and $LM$ is $\parallel$ $CF$ (both $\perp$ $AG$). $\therefore M$ is the mid-point of $AF$ and $LM = \frac{1}{2} CF$ (Th. 1.26, Cor. 1). $\therefore CF = BE + DG$.

(ii) The line passes through the $\square ABCD$. In the trapezoid $BEDG$, $L$ is the mid-point of $BD$ and $LM$ is $\parallel$ $BE$ and $DG$ (as shown above). $\therefore M$ is the mid-point of the diagonal $GE$. $\therefore LM = \frac{1}{2} (BE - DG)$ (Th. 1.29, Cor.). But, as shown above, also $LM = \frac{1}{2} CF$. $\therefore CF = BE - DG$.

1.18. $ABC$ is a triangle. $AD$ and $AE$ are two perpendiculums drawn from $A$ on the bisectors of the base angles of the triangle $B$ and $C$ respectively. Prove that $DE$ is $\parallel$ $BC$.

Construction: Produce $AD$ and $AE$ to meet $BC$ in $F$ and $G$ (Fig. 19).

![Figure 19](image)

Proof: $\triangle ADB$ and $FDB$ are congruent, since $AD$ is $\perp DB$ (Th. 1.11). $\therefore AD = DF$. Similarly, $\triangle AEC$ and $GEC$ are congruent (Th. 1.11). $\therefore AE = EG$. In the $\triangle AFG$, $D$ and $E$ are proved to be the mid-points of $AF$ and $AG$. $\therefore DE$ is $\parallel FG$ or $BC$ (Th. 1.26, Cor. 1).

1.19. $P$ and $Q$ are two points on either side of the parallel lines $AB$ and $CD$ so that $AB$ lies between $P$ and $CD$. Two points $L$ and $M$ are taken on $AB$ and $CD$. Find another two points $X$ and $Y$ on $AB$ and $CD$ so that $(PX + XY + YQ)$ is a minimum and $XY$ is $\parallel LM$.

Construction: From $Q$ draw $QR$ equal and $\parallel$ to $LM$; then join $PR$ to cut $AB$ in $X$. Draw $XY$ $\parallel$ $LM$ to meet $CD$ in $Y$. Hence $X$ and $Y$ are the two required points (Fig. 20).
FIGURE 20

Proof: \( RQ \) is equal and \( \parallel \) to \( LM \) and \( XY \parallel LM \), \( RQ =, \parallel XY \) (Th. 1.7). \( RQ \) is equal and \( \parallel \) to \( LM \) and \( XY \parallel LM \), \( \therefore XYQR \) is a parallelogram. \( YQ = XR \) and \( \therefore XY = LM \), hence the sum of \( (PX + XY + YQ) = PX + LM = XR = PR + LM \). To prove this is a minimum, let us take \( X'Y' \) any other \( \parallel \) to \( LM \) and join \( X'P, X'R \) and \( Y'Q \). \( X'Y' = LM \) (in \( \square X'Y'LM \)). Also \( X'Y'QR \) is a \( \square \). \( X'R = Y'Q \). But in the \( \triangle X'PR \), \( (PX' + X'R) > PR \) (Th. 1.18). \( \therefore X'Y' = LM \). \( \therefore (PX' + X'Y' + Y'Q) > PR + LM \) or \( (PX' + X'Y' + Y'Q) > (PX + XY + YQ) \). \( \therefore X' \) and \( Y' \) are the two required points.

1.20. Show that the sum of the two perpendiculars drawn from any point in the base of an isosceles triangle on both sides is constant.

Construction: Let \( ABC \) be the isosceles \( \triangle \) and \( D \) any point on its base \( BC \). \( DE, DF \) are the \( \perp \)s on \( AB \) and \( AC \). Draw \( BG \perp AC \), and \( DL \perp BG \) (Fig. 21).

FIGURE 21

Proof: The figure \( DLGF \) is a rectangle (construct). \( \therefore DF = LG \). Again, \( DL \parallel FG \) (both \( \perp BG \)). \( \therefore \angle BDL = \angle C \). \( \therefore \angle C = \angle B \) (since \( AC = AB \), Th. 1.12), \( \therefore \angle BDL = \angle L \). Also \( \angle BDE \)
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\[ \angle DLB = \text{right angle}. \] Hence \( \triangle s \ BDE \) and \( \triangle s \ BLD \) are congruent (Th. 1.11). \( \therefore DE = BL \). Therefore, \( DE + DF = BL + LG = BG \) = fixed quantity (since \( BG \) is \( \perp \) from \( B \) on \( AC \) and both are fixed).

1.21. Draw a straight line parallel to the base \( BC \) of the triangle \( ABC \) and meeting \( AB \) and \( AC \) (or produced) in \( D \), \( E \) so that \( DE \) will be equal to (i) the sum of \( BD \) and \( CE \); (ii) their difference.

CONSTRUCTION: Bisect the \( \angle B \) to meet the bisectors of the \( \angle C \) internally and externally in \( P \) and \( Q \). Draw from \( P \) and \( Q \) the parallels \( D_1PE_1 \) and \( QE_2D_2 \) to \( BC \) to meet \( AB \) and \( AC \) respectively in \( D_1 \), \( D_2 \) and \( E_1 \), \( E_2 \). Then \( E_1D_1 \) and \( E_2D_2 \) are the two required lines (Fig. 22).

Figure 22

Proof: (i) \( \because E_1D_1 \parallel BC \). \( \therefore \angle CBP = \angle BPD_1 \). But \( \angle CBP = \angle PBD_1 \) (construct). \( \therefore \angle BPD_1 = \angle PBD_1 \). \( \therefore BD_1 = D_1P \) (Th. 1.13). Similarly, \( CE_1 = E_1P \). Hence \( BD_1 + CE_1 = D_1P + E_1P = D_1E_1 \). \( \therefore E_1D_1 \) is the required first line.

(ii) In a similar way, \( \because QE_2D_2 \parallel BC \). \( \therefore \angle CBQ = \angle BQD_2 \).

But \( \angle CBQ = \angle QBD_2 \). \( \therefore \angle QBD_2 = \angle QBD_2 \). \( \therefore BD_2 = QD_2 \).

Likewise, \( CE_2 = QE_2 \). \( \therefore BD_2 - CE_2 = QD_2 - QE_2 = D_2E_2 \). \( \therefore D_2E_2 \) is the required second line.

1.22. \( ABCD \) is a parallelogram. From \( D \) a perpendicular \( DR \) is drawn to \( AC \). \( BN \) is drawn \( \parallel \) to \( AC \) to meet \( DR \) produced in \( N \). Join \( AN \) to intersect \( BC \) in \( P \). If \( DRN \) cuts \( BC \) in \( Q \), prove that (i) \( P \) is the mid-point of \( BQ \); (ii) \( AR = BN + RC \).

CONSTRUCTION: Draw \( BS \parallel AC \), then join \( CN \) (Fig. 23).

Proof: (i) \( \triangle s \ ABS, CDR \) are congruent (Th. 1.11). \( \therefore AS = CR \) and \( BS = DR \). Since the figure \( BSRN \) is a rectangle (\( BS \) and \( NR \) are both \( \perp \) \( AC \)), \( \therefore BS = NR \). \( \therefore DR = NR \). Hence \( \triangle s \ ADR, ANR \) are congruent (Th. 1.10). \( \therefore \angle DAR = \angle RAN \). But \( \angle DAR = \angle RCB \). \( \therefore \angle CBN \) = \( \angle CBN \) and \( \angle RAN = \angle PNB \) (since \( BN \) is \( \parallel \) \( AC \)). \( \therefore \angle CBN \)
FIGURE 23

\[ \angle PNB \]; \( PB = PN \). \( \therefore \angle BNQ \) = right angle, \( \therefore \) in the \( \triangle BNQ \), \( \angle PNQ = \angle PQN \) (complementary to equal \( \angle s \)). \( \therefore PQ = PN \).

Hence \( PB = PQ = PN \) or \( P \) is the mid-point of \( BQ \).

(ii) Since \( BSRN \) is a rectangle, \( \therefore BN = SR \). But \( AS = CR \) (as shown earlier). \( \therefore AR = AS + SR = CR + BN \).

1.23. \( \triangle ABC \) is an isosceles triangle in which the vertical angle \( A = 120^\circ \). If the base \( BC \) is trisected in \( D \) and \( E \), prove that \( \triangle ADE \) is an equilateral triangle.

Construction: Draw \( AF \) and \( DG \perp s \) to \( BC \) and \( AB \) (Fig. 24).

FIGURE 24

Proof: \( \therefore \angle A = 120^\circ \) in \( \triangle ABC \), \( \therefore \angle B + \angle C = 60^\circ \) (Th. 1.8). But \( \angle B = \angle C \) since \( AB = AC \) (Th. 1.12). \( \therefore \angle B = \angle C = 30^\circ \).

\( \triangle BDG \) is right angled at \( G \) and \( \angle B = 30^\circ \). \( \therefore DG = \frac{1}{2} BD \) (Th. 1.28). \( \therefore AF \perp BC \) and \( AB = AC \), \( \therefore BF = CF \) (Th. 1.12, Cor. 1).

Since \( BD = DE = CE \) (given), \( \therefore DF = FE \) or \( DF = \frac{1}{2} DE = \frac{1}{2} BD \). \( \therefore DG = DF \). \( \triangle ADG, ADF \) are congruent (Th. 1.15).

\( \therefore \angle DAG = \angle DAF \). Similarly, \( \angle EAC = \angle EAF \). But since \( \triangle s ADB, AEC \) are congruent (Th. 1.10), \( \angle DAB = \angle EAC \) and \( AD = AE \). Hence \( \angle DAB = \angle DAF = \angle EAF = \angle EAC = 30^\circ \).

Therefore, \( \angle DAE = 60^\circ \), but, since \( AD = AE \), then \( \triangle ADE \) is an equilateral triangle.
1.24. ABC and CBD are two angles each equal to 60°. A point O is taken inside the angle ABC and the perpendiculars OP, OQ, and OR are drawn from O to BA, BC, and BD respectively. Show that OR = OP + OQ.

Construction: From O draw the line FOG to make an angle 60° with AB or BC; then drop the two perpendiculars BL and GK from B and G on FOG and AB respectively (Fig. 25).

**Proof:** Since FOG makes with AB and BC angles = 60°, \( \triangle FBG \) is equilateral. According to Problem 1.20, \( OP + OQ = GK = BL \) (as the altitudes of an equilateral triangle are equal). \( \therefore \angle FGB = \angle GBD = 60° \), \( \therefore BD \parallel FG \). \( \therefore BROL \) is a rectangle. \( \therefore LB = OR \). \( \therefore OR = OP + OQ \).

1.25. Construct a right-angled triangle, given the hypotenuse and the difference between the base angles.

Construction: The hypotenuse BC and the difference of the base angles \( \angle E'AD' \) are given. Draw \( \angle EAD = \angle E'AD' \) and take \( AD = \frac{1}{2} BC \). Then drop \( DE \perp AE \) and produce \( DE \) from both sides to B and C so that \( DB = DC = \frac{1}{2} BC \), the given hypotenuse. Then \( ABC \) is the required triangle (Fig. 26).

**Proof:** According to Problem 1.3, the angle subtended by the altitude and median from the right-angled vertex to the hypotenuse equals the difference of the base angles. Also \( AD \) is half the hypotenuse (Th. 1.27). In the \( \triangle ABC \), \( DA = DB = DC = \frac{1}{2} \) given hypotenuse. \( \therefore \angle DAB = \angle DBA \) and \( \angle DAC = \angle DCA \). \( \therefore \angle DAB + \angle DAC = \angle BAC = \) right angle (half the sum of the angles of a \( \triangle \), Th. 1.8). Again, \( \angle EAD = (\angle C - \angle B) = \) given angle. \( \therefore ABC \) is the required \( \triangle \).
1.26. If the three medians of a triangle are known, construct the triangle.

Construction: Draw the $\triangle CDG$ having $CD$, $DG$, and $CG$ equal to $C'D'$, $A'F'$ and $B'E'$ the given medians. Draw the medians $CJ$ and $GK$ in the $\triangle CDG$ to intersect in $F$. From $F$ draw $FA \parallel$ and equal to $DG$ or the given $F'A'$. Join $AD$ and produce it to $B$, so that $AD = DB$. Then $ABC$ is the required $\triangle$ (Fig. 27).

Proof: Join $DF$. In the $\triangle CDG$, $F$ is the point of concurrency of its medians. Hence $FJ = \frac{1}{2} CF$ and $FK = \frac{1}{2} FG$ (Th. 1.34). Since $FA$ is $\parallel$ and equal to $DG$, $\therefore ADGF$ is a parallelogram (Th. 1.23). $\therefore FG$ is $\parallel$ and equal to $AD$ and $DB$ (Th. 1.22). Again, $DBGF$ is also a parallelogram. Therefore, $DG$ and $FB$ bisect one another (Th. 1.22). $\therefore FJ$
= JB = \frac{1}{2} FB. \therefore CF = FB or F is the mid-point of BC. But L is the intersection of the medians CD and AF. \therefore BE is also the third median (Th. 1.34). \therefore DF is || and = CE (Th. 1.26, Cor. 1) and also is || and = BG, \therefore BG is || and equal to CE. Hence BGCE is another parallelogram. \therefore BFC is one diagonal and BE = CG. Therefore, the medians of the triangle ABC are equal to the sides of angle CDG or equal to the given medians. Hence ABC is the required triangle.

1.27. ABC is a triangle. On AB and AC as sides, two squares ABDE and ACFG are drawn outside the triangle. Show that CD, BF, and the perpendicular from A on BC meet in one point.

**Construction:** Draw BP \perp CD. BP produced meets HA (the perpendicular from A on BC) produced in R. Join RC cutting BF in Q (Fig. 28).

**Proof:** \therefore BP is \perp CD, \therefore \angle ABP = \angle BDP (since both are complementary to \angle PBD). \therefore \angle BAE = \text{right angle}, \therefore \angle RAE + \angle BAH = \text{right angle} (Th. 1.1). But since AH is \perp BC, \therefore in the \triangle AHB, \angle BAH + \angle ABH = \text{right angle}. \therefore \angle RAE = \angle ABH, since, in the square ABDE, \angle BAE = \angle ABD = \text{right angle}. By adding, \angle RAE + \angle BAE = \angle ABH + \angle ABD or \angle RAB = \angle CBD. Since also AB = BD, \therefore \triangle ABR and BDC are congruent (Th. 1.11). \therefore AR = BC. Again, \angle RAG = \angle CAH (since both are complementary to \angle CAH), and, since AC = CF in the square ACFG, then the \triangle ARC and CBF are congruent (Th. 1.10). \therefore \angle ACR = \angle CFB. \therefore \angle ACR + \angle RCF = \angle ACF = \text{right angle in the square AF, \angle CFB + \angle RCF or \angle QCF = \text{right angle}.} \therefore \text{In the \triangle CQF, \angle CQF = \text{right angle or CQ is \perp BF.} Now, in the \triangle BRC,
RH, CP, and BQ are the altitudes to the sides. Therefore, they are concurrent (Th. 1.33); i.e., AH, CD and BF meet in one point.

1.28. If any point D is taken on the base BC of an isosceles triangle ABC and DEF is drawn perpendicular to the base BC and meets AB and AC or produced in E and F, show that (DE + DF) is a constant quantity and equals twice the perpendicular from A to BC.

Construction: Draw AP and AQ \perp s BC and DEF respectively. From B draw BF' \perp BC to meet CA produced in F' (Fig. 29).

\textbf{Figure 29}

\textit{Proof}: AP is \perp BC in the isosceles \triangle ABC. \therefore AP bisects \angle A (converse Th. 1.12, Cor. 1). But DEF is \parallel AP (both are \perp BC). \therefore \angle BAP = \angle AEF and \angle CAP = \angle AFE (Th. 1.6). Since \angle BAP = \angle CAP, \therefore \angle AEF = \angle AFE. \therefore AE = AF (in the \triangle AEF, Th. 1.13). But AQ is \perp EF (construct). \therefore the \triangle AEQ and AFQ are congruent (Th. 1.11). \therefore EQ = QF. Hence (DE + DF) = 2(DE + EQ) = 2DQ. Since AQDP is a rectangle, \therefore DQ = AP. Therefore, (DE + DF) = 2AP = constant. The extreme case is when D and E approach the extremities of the base until they coincide with B, where obviously D'F' = 2D'Q' = 2AP = constant.

1.29. Any line is drawn through O, the point of concurrence of the medians of a triangle ABC. From A, B, and C three perpendiculars AP, BQ, and CR are drawn to this line. Show that AP = BQ + CR.

Construction: Let AD and CE be two medians of the \triangle ABC. From D and G, the mid-points of BC, AO drop DF and GN \perp s to the straight line ROQ (Fig. 30).
Proof: The figure $BQRC$ is a trapezoid and $D$ is the mid-point of $CB$. Since $DF$ is $\parallel BQ$ and $CR$ (\(\perp\)s on line $ROQ$), \(\therefore\) $F$ is the mid-point of $RQ$ (Th. 1.25) and $DF = \frac{1}{2} (BQ + CR)$ (Th. 1.26, Cor. 3). Now, since $O$ is the point of concurrency of the medians of the $\triangle ABC$, then $AO = \frac{2}{3} AD$ (Th. 1.34). \(\therefore\) $AO = 2 OD$. \(\therefore\) $GO = OD$. But $DF$, $GN$ are \(\perp\)s to $ROQ$. Hence the two opposite $\triangle$s $ODF$ and $OGN$ are congruent (Th. 1.11). \(\therefore\) $DF = GN$. Again, in the $\triangle AOP$, $G$ is the mid-point of $AO$ and $GN$ is $\parallel AP$ (\(\perp\)s to $ROQ$). \(\therefore\) $N$ is the mid-point of $OP$ and $GN = \frac{1}{2} AP$ (Th. 1.26, Cor. 1). Therefore, $AP = BQ + CR$.

1.30. Construct an isosceles trapezoid having given the lengths of its two parallel sides and a diagonal.

Analysis: Suppose $ABCD$ is the required trapezoid (Fig. 31). If through $C$, $CE$ is drawn $\parallel AD$, $AEC$ is a $\square$, \(\therefore\) $AE = DC = \text{given length}$. Also $CE = AD = CB$. \(\therefore\) $C$ lies on the straight line which bisects $EB$ at right angles. But $C$ also lies on the circle with center $A$ and radius equal to the given diagonal. Hence,
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SYNTHESIS: From $AB$ the greater of the two given sides, cut off $AE$ equal to the less; bisect $EB$ in $F$; from $F$ draw $FG \perp EB$. With center $A$ and radius equal to the given diagonal, describe a circle cutting $FG$ in $C$. Join $CE$, $CB$, and complete the $\square$ $ED$. $ABCD$ will be the required trapezoid.

Proof: Since $DE$ is a $\square$, \( DC = AE \) = given length.

Also $DA = EC = CB$ (from congruency of $\triangle$s $BFC$, $EFC$, Th. 1.10) and $AB$, $AC$ are by construction the required lengths.

1.31. On the sides of a triangle $ABC$, squares $ABDE$, $ACFG$, $BCJK$ are constructed externally to it. $BF$, $AJ$, $CD$, $AK$ are joined. If $FC$, $DB$ are produced to meet $AJ$, $AK$ in $H$, $I$ respectively and $X$, $Y$ are the intersections of $BF$, $CD$ with $AC$, $AB$ respectively, show that $X$, $H$ are equidistant from $C$ and $Y$, $I$ are also equidistant from $B$. Prove also that the perpendiculars from $A$, $B$, $C$ on $GE$, $DK$, $FJ$ respectively intersect in the centroid of the triangle $ABC$ and that $GE$, $DK$, $FJ$ are respectively double the medians from $A$, $B$, $C$.

\[\text{Figure 32}\]

Proof: (i) $\triangle$s $ACJ$, $FCB$ are congruent (Th. 1.10) (Fig. 32). \( \therefore \angle CAJ = \angle CFB \). Since $\angle CHJ = \text{right angle} + \angle CAH$ and $\angle CXB = \text{right angle} + \angle CFB$, \( \therefore \angle CHJ = \angle CXB \). Hence $\triangle$s $ACH$, $FCX$ are congruent (Th. 1.11). \( \therefore CX = CH \). Similarly, $BY = BI$.

(ii) Let $AP$, $BQ$, $CR$ be the $\perp$s from $A$, $B$, $C$ on $GE$, $DK$, $FJ$ respectively. Produce $PA$ to meet $BC$ in $L$ and draw $BM$, $CN \perp$ $PAL$. Then $\triangle$s $APE$, $BMA$ are congruent (Th. 1.11). \( \therefore AP = BM \). Similarly, $\triangle$s $APG$, $CNA$ are congruent. \( \therefore AP = CN = BM \). Also $\triangle$s $BML$, $CNL$ are congruent. \( \therefore BL = CL \). Hence $PAL$ is a median.
in \( \triangle ABC \). Similarly, \( QB, RC \) produced are the other two medians of \( \triangle ABC \). Therefore, they meet at the centroid. Again, from the congruence of \( \triangle APE, BMA \), \( \therefore PE = MA \). Also, \( PG = NA \). \( \therefore GE = MA + NA = 2(AN + LN) = 2AL \).

1.32. The point of concurrence \( S \) of the perpendiculars drawn from the middle points of the sides of a triangle \( ABC \), the orthocenter \( O \), and the centroid \( G \) are collinear and \( OG = 2SG \).

**Construction:** Let \( D, E \) be the mid-points of \( BC, AB \). Bisect \( AO, CO, AG, GO \) in \( H, K, L, M \). Join \( DE, HK, LM \) (Fig. 33).

**Figure 33**

Proof: \( HK \) is \( || \) and \( = ED \) (each being \( || AC \) and \( \frac{1}{2} AC \) in \( \triangle AOC, ABC \), Th. 1.26). \( \triangle ESD, KOH \) have their corresponding angles equal. Since their sides are respectively \( || \), \( \therefore \) they are congruent and \( SD = HO \) (Th. 1.11). \( \therefore LM = \frac{1}{2} AO = HO \), \( \therefore \) in \( \triangle AGO \), since \( L, M \) are the mid-points of \( AG, GO \), \( \therefore SD = LM = \frac{1}{2} \). But \( G \) is the centroid. \( \therefore GD = \frac{1}{2} AG = LG \) (Th. 1.34) and \( \angle SDG = \angle GLM \) (since \( SD \) is \( || LM \), being \( || AO \)). \( \therefore \) \( \triangle SDG, MLG \) are congruent (Th. 1.10). \( \therefore \) \( \angle SGD = \angle MGL \). But they are vertically opposite at \( G \). \( \therefore SG \) is in line with \( GM \) or \( GO \) (converse, Th. 1.3). Also, \( SG = MG \) or \( OG = 2SG \). The line \( OGS \) referred to is the **Euler line**.

**Miscellaneous Exercises**

1. Two triangles are congruent if two sides and the enclosed median in one triangle are respectively equal to two sides and the enclosed median of the other.

2. The two sides \( AB, AC \) in the triangle \( ABC \) are produced to \( D, E \) respectively so that \( BD = BC = CE \). If \( BE \) and \( CD \) intersect in \( F \), show that \( \angle BFD = \text{right angle} - \frac{1}{2} \angle A \).
3. From any point $D$ on the base $BC$ of an isosceles triangle $ABC$, a perpendicular is drawn to cut $BA$ and $CA$ or produced in $M$ and $N$. Prove that $AMN$ is an isosceles triangle.

4. $ABC$ is a triangle in which $AB$ is greater than $AC$. If $D$ is the middle point of $BC$, then the angle $CAD$ will be greater than the angle $BAD$.

5. The straight line drawn from the middle point of the base of a triangle at right angles to the base will meet the greater of the two sides, not the less.

6. $ABC$ is a right-angled triangle at $A$. The altitude $AD$ is drawn to the hypotenuse $BC$. $DA$ and $CB$ are produced to $P$ and $Q$ respectively so that $AP = AB$ and $BQ = AC$. Show that $CP = AQ$.

7. A square is described on the hypotenuse $BC$ of a right-angled triangle $ABC$ on the opposite side to the triangle. If $M$ is the intersection of the diagonals of the square and $LMN$ is drawn perpendicular to $MA$ to meet $AB$, $AC$ produced in $L$, $N$ respectively, then $BL = AC$, $CN = AB$.

8. Show that the sum of the altitudes of a triangle is less than the sum of its three sides.

9. On $BC$ as a base, an equilateral triangle $ABC$ and an isosceles triangle $DBC$ are drawn on the same side of $BC$ such that $\angle D = \text{half } \angle A$. Prove that $AD = BC$.

10. $P$ is any point inside or outside the triangle $ABC$. $AP$, $BP$, $CP$ are produced to $R$, $S$, $T$ respectively so that $AP = PR$, $BP = PS$, $CP = PT$. Show that the triangles $RST$ and $ABC$ are equiangular.

11. The interior and exterior angles at $C$ of a triangle $ABC$ are bisected by $CD$, $CF$ to meet $AB$ and $BA$ produced in $D$, $F$ respectively. From $D$ a line $DR$ is drawn parallel to $BC$ to meet $AC$ in $R$. Show that $FR$ produced bisects $BC$. (Produce $DR$ to meet $CF$ in $S$.)

12. $ABC$ is an isosceles triangle in which $AB = AC$. On $AB$ a point $G$ is taken and on $AC$ produced the distance $CH$ is taken so that $BG = CH$. Prove that $GH > BC$.

13. Construct a triangle having given the base, the difference of the base angles, and the difference of the other two sides. (Use Problem 1.10).

14. Show that the sum of the three medians in a triangle is less than its perimeter and greater than $\frac{3}{2}$ the perimeter.

15. $ABC$ is an isosceles triangle and $D$ any point in the base $BC$; show that perpendiculars to $BC$ through the middle points of $BD$ and $DC$ meet $AB$, $AC$ in points $H$, $K$ respectively so that $BH = AK$ and $AH = CK$. (Join $DH$, $DK$.)

16. The side $AB$ of an equilateral triangle $ABC$ is produced to $D$ so that $BD = 2AB$. A perpendicular $DF$ is drawn from $D$ to $CB$ produced. Show that $FAC$ is a right angle.

17. On the two arms of a right angle with vertex at $A$, $AB$ is taken $= AD$ and also $AC = AE$, so that $B$, $C$ are on the same area of $\angle A$. Prove that the perpendicular from $A$ to $CD$ when produced bisects $BE$. 

18. \(ABC\) is an obtuse angle and \(AB = 2 \ BC\). From \(G\), the middle point of \(AB\), a perpendicular is drawn to it and from \(C\) another perpendicular is drawn to \(CB\) to meet the first one in \(H\). Show that \(\angle AHG = \frac{1}{3} \angle AHC\).

19. Show that the sum of the three bisectors of the angles of a triangle is greater than half its perimeter.

20. Construct a triangle, having given the base, the vertical angle, and (a) the sum; (b) the difference of the sides.

21. \(ABC\) is a triangle. On \(AB\) a point \(M\) is taken so that \(AM = \frac{1}{3} \ AB\). \(BC\) is bisected in \(N\) and \(AN\) and \(CM\) intersect in \(R\). Show that \(R\) is the middle of \(AN\) and that \(MR = \frac{1}{4} \ MC\). (Draw \(NP \parallel MC\).)

22. A triangle \(ABC\) is turned about the vertex \(A\) into the position \(AB'C'\); if \(AC\) bisects \(BB'\), prove that \(AB'\) (produced, if necessary) will bisect \(CC'\).

23. \(ABCDEF\) is a hexagon. Prove that its perimeter > \(\frac{1}{3} \ (AD + BE + CF)\).

24. Any point \(D\) is taken in \(AB\) one of the equal sides of an isosceles triangle \(ABC\); \(DEF\) is drawn meeting \(AC\) produced in \(F\) and being bisected by \(BC\) in \(E\). Show that \(CF = BD\).

25. \(ABCD\) is a quadrilateral in which \(AB = CD\) and \(\angle C > \angle B\). Prove that \(DB > AC\) and \(\angle A > \angle D\).

26. Prove that the interior angle of a regular pentagon is three times the exterior angle of a regular decagon. (Use Cor., Th. 1.9.)

27. The vertical angle \(A\) in an isosceles triangle \(ABC\) is half a right angle. From \(A\), the altitude \(AD\) is drawn to the base \(BC\). If the perpendicular from \(C\) to \(AB\) cuts \(AD\) in \(P\) and meets \(AB\) in \(Q\), show that \(PQ = BQ\).

28. \(A, B, C\) are three points on a straight line. On \(AB\) and \(AC\) squares \(ABDE, ACFG\) are described so as to lie on the same side of the straight line. Show that the straight line through \(A\) at right angles to \(GB\) bisects \(EC\).

29. Construct a triangle, having given one of its sides and the point of concurrence of its medians.

30. The sum of the distances (perpendiculars) of the vertices of a triangle on any straight line is equal to the sum of the distances of the midpoints of the sides of the triangle on the same line. (Use Problems 1.17 and 1.29.)

31. The vertical angle \(A\) of an isosceles triangle \(ABC\) is \(\frac{1}{4}\) of each of the base angles. Two points \(M, N\) are taken on \(AB, AC\) respectively so that \(BM = BC = CN\). If \(BN\) and \(CM\) intersect in \(D\), show that \(\angle MDN = \angle B\). (See Problem 1.5.)

32. If any two points \(F, G\) are taken inside an acute angle \(BAC\), find two points \(P\) on \(AB\) and \(Q\) on \(AC\), so that the sum of \(FP + PQ + QG\) will be a minimum.

33. If a triangle and a quadrilateral are drawn on the same base and the quadrilateral is completely inside the triangle, show that the perimeter of the triangle > the perimeter of the quadrilateral.
34. *P* is any point in *AB*, one of the shorter sides of a given rectangle *ABCD*. Show how to construct a rhombus having one of its angular points at *P*, and with its other angular points one on each of the other sides of the rectangle.

35. *ABCDE* is an irregular pentagon. Prove that if each pair of its sides, when produced, meet in five points, the sum of the five resulting angles will be equal to \(\frac{3}{2}\) the sum of the angles of the pentagon.

36. *ABC* is a triangle in which the angle \(B = 120^\circ\). On *AC* at the opposite side of the triangle, an equilateral triangle *ACD* is described. Show that *DB* bisects \(\angle B\) and equals \((AB + BC)\).

37. *ABCD* is a parallelogram. If the two sides *AB* and *AD* are bisected in *E*, *F* respectively, show that *CE*, *CF*, when joined, will cut the diagonal *BD* into three equal parts.

38. *A*, *B*, *C* are three given points not on the same straight line. Draw a line to pass through *A* so that if two perpendiculars are drawn to it from *B* and *C*, then the one from *C* will be double that from *B*.

39. *ABC* is a right-angled triangle at *B*. On *AB* and *BC* two squares *ABDE*, *BCGH* are described outside the triangle. From *E*, *G* two perpendiculars *EL*, *GK* are drawn to *AC* produced. Show that \(AC = EL + GK\).

40. *D* is the middle point of the base *BC* of a triangle *ABC*. Prove that if the vertical angle *A* is acute, then \(6AD > \text{the perimeter of the triangle}\).

41. The bisectors of the angles of any quadrilateral form a second quadrilateral, the opposite angles of which are supplementary. If the first quadrilateral is a parallelogram, the second is a rectangle the diagonals of which are parallel to the sides of the parallelogram and equal to the difference of its adjacent sides. If the first quadrilateral is a rectangle, the second is a square.

42. The straight line *AB* is trisected in *C* so that *AC* is double *BC* and parallel lines through *A*, *B*, *C* (all on the same side of *AB*) meet a given line in *L*, *M*, *N*. Prove that \(AL\) with twice \(BM\) is equal to three times \(CN\).

43. Show that the distance (perpendicular) of the centroid of a triangle from a straight line is equal to the arithmetic mean of the distances of its vertices from this line. (Use Problem 1.17 and Exercises 30 and 42.)

44. Construct a triangle having given the positions of the middle points of its three sides.

45. *ABC* is a triangle in which \(AC > AB\) and *D* is the middle point of the base *BC*. From *D* a straight line *DFG* is drawn to cut *AC* in *F* and *BA* produced in *G* so that \(\angle AFG = \angle AGF\). Prove that *AF* is equal to half the difference between *AC* and *AB*. (Draw *AM \parallel DFG* and drop \(BM \perp AM\). Produce *BM* to meet *AC* in *N*. Join *MD*. Use Problem 1.10.)

46. Show that the sum of the perpendiculars from any point inside an equilateral triangle on its sides is constant and equal to any altitude in the triangle. (From the point draw a line \(\parallel\) to any side and use Problem 1.20.)
47. ABC is an equilateral triangle. If the two angles B, C are bisected by BD, CD and from D two parallels DR, DQ are drawn to AB, AC to meet BC in R, Q, show that R, Q are the points of trisection of BC.

48. AD, BF, CG are the three medians of a triangle ABC. AR, BR are drawn parallel to BF, AC and meet in R. Show that R, G, F are collinear and that RC bisects DG.

49. In a triangle ABC, AB is greater than AC. Find a point P on BC such that \( AB - AP = AP - AC \). [Apply Problem 1.10, part (ii).]

50. Construct a triangle having given the base, one of the base angles, and the difference between the other two sides.

51. ABC is an isosceles triangle having \( \angle A = 45^{\circ} \). If AD, CF, the two altitudes from A, C on BC, AB respectively, meet in G, prove that AC = FG = CF.

52. The base angle in an isosceles triangle ABC is three times the vertical angle A. D is any point on the base BC. On CA the distance CF is taken to CD. If FD is joined and produced to meet AB produced in G, prove that the bisector of the external angle C \( \parallel FDG \) and that FD + FG > AB. (Use Problem 1.5.)

53. ABC is a right-angled triangle at A. Produce BA and CA to X, Y so that AX = AC, AY = AC. If XY is bisected in M, show that MA produced is perpendicular to BC.

54. Construct a triangle having given its perimeter and two base angles.

55. On AB and BC of a triangle ABC, two squares ABDE, BCJK are constructed outside the triangle. Prove that CD \( \perp AK \).

56. Prove that any angle of a triangle is either acute, right, or obtuse, according to whether the median from it to the opposite side is greater than or equal or less than half this side. State and prove the opposite of this problem.

57. From a given point P draw three straight lines PA, PB, PC of given lengths such that A, B, C will be on the same line and AB = BC.

58. ABC is a triangle and AC > AB. From A two straight lines AD, AG are drawn to meet the base in D, G so that \( \angle DAC = \angle B \) and \( \angle GAB = \angle C \). Show that DC > BG.

59. Prove by the methods of Chapter 1 that if D, E, F are the feet of the perpendiculars from A, B, C respectively on the opposite sides of an acute-angled triangle ABC, then AD, BE, CF bisect the angles of the pedal triangle DEF.

60. In Exercise 59, if DG, DH are drawn perpendicular to AC, AB and G, H are joined, prove that GH is equal to half the perimeter of the pedal triangle DEF. (Produce DG, DH to meet EF in L, M.)

61. Construct a triangle, having given the feet of the perpendiculars on the sides from the opposite vertices.
62. **ABC** is an isosceles triangle in which the vertical angle $A = 20^\circ$. From $B$, a straight line $BD$ is drawn to subtend a $60^\circ$ angle with $BC$ and meet $AC$ in $D$. From $C$ another straight line $CF$ is drawn to subtend a $50^\circ$ angle with $BC$ and meet $AB$ in $F$. $DF$ is joined and produced to meet $CB$ produced in $G$. Show that $BD = BG$. (Draw $DR \parallel BC$ to meet $AB$ in $R$. Join $RC$ cutting $DB$ in $M$, and join $MF$.)

63. Prove that if pairs of opposite sides of a quadrilateral are produced to meet in two points, then the bisectors of the two angles at these two points will subtend an angle which is equal to half the sum of one pair of opposite angles in the quadrilateral.

64. $AB$, $AC$ are two straight lines intersecting in $A$. From $D$, any point taken on $AC$, a straight line $DG$ is drawn parallel to the bisector of the angle $A$. If $F$ is any point on $DG$, show that the difference between the perpendiculars drawn from $F$ on $AB$, $AC$ is fixed.

65. The side $AB$ in a triangle $ABC$ is greater than $AC$. If the two altitudes $BD$, $CF$ are drawn, show that $BD > CF$.

66. Draw a straight line to subtend equal angles with two given intersecting lines and bear equal distances from two given points. (From the midpoint of the line joining the two points, draw $\perpendicular$ the bisector of the angle between the straight lines.)

67. Construct a right-angled triangle, having given the hypotenuse and the altitude from the right vertex on the hypotenuse.

68. In any triangle the sum of two medians is greater than the third median, and the median bisecting the greater side is less than the median bisecting the smaller side.

69. Show that the bisector of any angle of a triangle is less than half the sum of the two surrounding sides.

70. $ABCD$ is a quadrilateral and $XY$ is any straight line. Show that if $M$ is the intersection of the lines joining the mid-points of the opposite side, $AB$, $CD$ and $BC$, $AD$, then the sum of the distances of $A$, $B$, $C$, $D$ from $XY$ will be equal to four times the distance of $M$ from $XY$.

71. A point is moving on a hypotenuse of a given right-angled triangle. Find the location of this point such that the line joining the feet of the perpendiculars from it on the sides of the triangle is minimum.

72. $ABCD$ is a parallelogram. On $BC$ another parallelogram $BCA'D'$ is described so that $AB$, $BD'$ will be adjacent sides. A third parallelogram $ABD'C'$ is constructed. Show that $AA'$, $CC'$, $DD'$ are concurrent.

73. $ABC$ is a right-angled triangle at $A$. $D$ is the middle point of the hypotenuse $BC$. Join $AD$ and produce it to $E$, so that $AD = DE$. If the perpendicular from $E$ on $BC$ meets the bisectors of the angles $B$, $C$ or produced in $F$, $G$, show that (a) $EB = EF$; (b) $EC = EG$.

74. Construct a right-angled triangle, having given (a) one of its sides and the sum of the hypotenuse and second side; (b) one of its sides and the difference between the hypotenuse and the second side.
75. $ABCD$, $EBFD$ are a square and a rectangle having the same diagonal $BD$. If $A, E$ are on the same side of $BD$, then $AG$ is drawn perpendicular to $AE$ to meet $BE$ (or produced) in $G$. Prove that $BG = ED$.

76. $ABC$ is a triangle and $AE$, $BM$, $CN$ are its three medians. Produce $AE$ to $D$, so that $AE = ED$ and $BC$ from both sides to $F$, $G$, so that $FB = BC = CG$. Show that the perimeter of each of the triangles $ADF$, $ADG$ is equal to twice the sum of the medians of the triangle $ABC$.

77. The point of intersection of the diagonals of the square described on the hypotenuse of a right-angled triangle is equally distant from the sides containing the right angle.

78. $O$ is a point inside an equilateral triangle $ABC$. $OA$, $OB$, $OC$ are joined and on $OB$ is described, on the side remote from $A$, an equilateral triangle $OBD$. Prove that $CD$ is equal to $OA$.

79. Construct a quadrilateral having given the lengths of its sides and of the straight line joining the middle points of two of its opposite sides.

80. In a triangle $ABC$, $AC$ is its greatest side. $AB$ is produced to $B'$ so that $AB'$ is equal to $AC$; $CB$ is produced to $B''$ so that $CB''$ is equal to $CA$: $CB'$, $CB''$ meet in $D$. Show that if $AB > BC$, then $AD > CD$.

81. $ABCD$ is a parallelogram having $AB = 2 BC$. The side $BC$ is produced from both sides to $E$ and $F$ so that $BE = BC = CF$. Show that $AF$ is $\perp DE$. (Join $LG$ and prove that $AGLD$ is a rhombus.)

82. Show that the centers of all the parallelograms which can be inscribed in a given quadrilateral so as to have their sides parallel to the diagonals of the quadrilateral lie in a straight line.

83. $D$, $E$, $F$ are the middle points of the sides $BC$, $CA$, $AB$ of a triangle $ABC$. $FG$ is drawn parallel to $BE$ meeting $DE$ produced in $G$. Prove that the sides of the triangle $CFG$ are equal to the medians of the triangle $ABC$.

84. On the halves of the base of an equilateral triangle, equilateral triangles are described remote from the vertex. If their vertices are joined to the vertex of the original triangle, show that the base is trisected by these lines. (From the vertices draw $\perp$s to the base.)

85. Which of the triangles that have the same vertical angle has the least perimeter, if two of its sides coincide with the arms of the given angle? Construct such a triangle if the perimeter is given. (Let $A$ be the given angle. On one of its arms, take $AD = \frac{1}{2}$ the given perimeter, then complete the isosceles $\triangle ADE$. The bisectors of $\angle s A$, $D$ meet in $G$. Draw $BGC \parallel DE$ to meet $AD$, $AE$ in $B, C$. Therefore, the isosceles $\triangle ABC$ is the required triangle with the least given perimeter, since $BG + BA = BD + BA = \frac{1}{2}$ the given perimeter, while $G$ is the mid-point of $BC$. To show that $\triangle ABC$ has the least perimeter, take $BB'$ on $AB = CC'$ on $CE$. Join $B'C'$, then $B'C' > BC$ according to Exercise 12. But, since $AB' + AC' = AB + AC$, then perimeter of $\triangle AB'C' >$ perimeter of $\triangle ABC$.)
86. If the bisectors of two angles of a triangle are equal, then the triangle is isosceles.

87. If two straight lines are drawn inside a rectangle parallel to one of the diagonals and at equal distances from it, then the perimeter of the parallelogram formed by joining the nearer extremities of these two lines will be constant.

88. Describe the quadrilateral $ABCD$, given the lengths of the sides in order and the angle between the two opposite sides $AB, CD$. (Through $B$ draw $BE$ parallel and equal to $CD$.)

89. If the two sides of a quadrilateral are equal, these sides being either adjacent or opposite, the line joining the middle points of the other sides makes equal angles with the equal sides.

90. Let $O$ be the middle point of $AB$, the common hypotenuse of two right-angled triangles $ACB$ and $ADB$. From $C, D$ draw straight lines at right angles to $OC, OD$ respectively to intersect at $P$. Show that $PC = PD$.

91. The perpendicular from any vertex of a regular polygon, having an even number of sides, to the straight line joining any two other vertices passes through a fourth vertex of the polygon.

92. $D$ is the middle point of the base $BC$ of an isosceles triangle $ABC$ and $E$ the foot of the perpendicular from $D$ on $AC$. Show that the line joining the middle point of $DE$ to $A$ is perpendicular to $BE$.

93. In Problem 1.31, if $AS, BT, CV$, the medians from $A$ to $GE, B$ to $DK, C$ to $FJ$ in the triangles $AGE, BDK, CFJ$, are produced inside the triangle $ABC$, they will meet at the orthocenter of the triangle $ABC$ and that $CB, CA, AB$ will be double $AS, BT, CV$ respectively. (Apply a similar procedure to that of Problem 1.31 as illustrated.)

94. The exterior angles of the triangle $ABC$ are bisected by straight lines forming a triangle $LMN$; $L, M, N$ being respectively opposite $A, B, C$. If $P, Q, R$ be the orthocenters of the triangles $LBC, MCA, NAB$ respectively, show that the triangle $PQR$ has its sides equal and parallel to those of $ABC$. [Let the bisectors of the interior angles of $\triangle ABC$ meet in $O$ (Th. 1.32). Therefore, $OA, OB, OC$ are $\perp s MN, NL, LM$ (Th. 1.1, Cor. 2). Consequently, the figures $ARBO, AQCO$ are $\square$s. $\therefore RB, AO, QC$ are equal and $\parallel$. $\therefore RBCQ$ is a $\square$. $\therefore RQ, BC$ are equal and $\parallel$. Similarly with $PQ, AB$ and $RP, AC$.

95. Show that the perpendiculars from the middle points of the sides of any triangle to the opposite sides of its pedal triangle are concurrent. (Join any two vertices of the pedal triangle to the mid-point of the opposite side of the original triangle and prove these lines are equal.)
CHAPTER 2
AREAS, SQUARES, AND RECTANGLES

Theorems and Corollaries

Areas of Polygons

2.35. Parallelograms on the same base and between the same parallels, or of the same altitude, are equal in area.

Corollary. Parallelograms on equal bases and between the same parallels, or of the same altitude, are equal in area.

2.36. The area of a parallelogram is equal to that of a rectangle whose adjacent sides are equal to the base and altitude of the parallelogram respectively.

2.37. If a triangle and a parallelogram are on the same base and between the same parallels, or of the same altitude, the area of the triangle is equal to half that of the parallelogram.

Corollary. If a triangle and a parallelogram are on equal bases, and between the same parallels or of the same altitude, the area of the triangle is equal to half that of the parallelogram.

2.38. The area of a triangle is equal to half that of a rectangle whose adjacent sides are respectively equal to the base and altitude of the triangle.

2.39. Triangles on the same, or on equal bases, and between the same parallels or of the same altitude, are equal in area.

2.40. Triangles of equal area, which are on equal bases in the same straight line and on the same side of it, are between the same parallels.

Corollary. Triangles of equal area, on the same base, and on the same side of it are between the same parallels.

2.41. Triangles of equal area, on the same, or on equal bases, are of the same altitude.

2.42. In every parallelogram, each diagonal bisects its area into two equal triangles.
Squares and Rectangles Related to Lines and Triangles

2.43. The squares on equal straight lines are equal in area, and conversely equal squares are on equal straight lines.

Theorems 2.44 to 2.50. The following identities have been proved true geometrically.

2.44. \( x(a + b + c) = xa + xb + xc \).

2.45. \( (a + b)^2 = a(a + b) + b(a + b) \).

2.46. \( a(a + b) = a^2 + ab \).

2.47. \( (a + b)^2 = a^2 + 2ab + b^2 \).

2.48. \( (a + b)(a - b) = a^2 - b^2 \).

2.49. \( (a - b)^2 = a^2 - 2ab + b^2 \).

2.50. \( (a + b)^2 - (a - b)^2 = 4ab \).

2.51. In any right-angled triangle, the square on the hypotenuse is equal to the sum of the squares on the sides containing the right angle. This is called Pythagoras's theorem.

Corollary 1. If \( N \) is any point in a straight line \( AB \), or in \( AB \) produced, and \( P \) any point on the perpendicular from \( N \) to \( AB \), then the difference of the squares on \( AP, BP \) is equal to the difference of the squares on \( AN, BN \).

Corollary 2. The converse of Corollary 1 is true.

2.52. If the square on one side of a triangle is equal to the sum of the squares on the other sides, then the angle contained by these sides is a right angle.

2.53. In an obtuse-angled triangle, the square on the side opposite the obtuse angle is greater than the sum of the squares on the sides containing the obtuse angle, by twice the rectangle contained by either of these sides and the projection, on this side produced, of the other side adjacent to the obtuse angle.

2.54. In any triangle, the square on the side opposite an acute angle is less than the sum of the squares on the sides containing the acute angle, by twice the rectangle contained by either of these sides and the projection on it of the other side adjacent to the acute angle.

2.55. In any triangle, the sum of the squares on two sides is equal to twice the square on half the base, together with twice the square on the median which bisects the base.

Corollary 1. If a straight line \( BC \) is bisected at \( D \) and \( A \) is any point in \( BC \), or \( BC \) produced, then the sum of the squares on \( AB \) and \( AC \) is equal to twice the sum of the squares on \( BD \) and \( AD \).

Corollary 2. The sum of the squares on the sides of a parallelogram is equal to the sum of the squares on its diagonals.
Solved Problems

2.1. If two equal triangles are on the same base and on opposite sides of it, the straight line joining their vertices is bisected by their common base, produced if necessary. Conversely, if the straight line joining the vertices of two triangles on the same base and on opposite sides of it be bisected by their common base or base produced, then the two triangles are equal in area.

(i) Let $ABC$, $BCD$ be the equal triangles. Join $AD$ meeting $BC$ in $E$. $AE$ will be equal to $ED$.

**Construction:** Draw $BE \parallel AC$ and $CF \parallel AB$. Join $FD$, $AF$ (Fig. 34).

![Figure 34](image)

**Proof:** $ACFB$ is a $\square$. $\therefore AG = GF$. $\therefore \triangle BFC = \triangle ABC$ (Th. 1.22) and $\triangle BDC = \triangle ABC$ (hypothesis). $\therefore \triangle BFC = \triangle BDC$. $\therefore FD$ is $\parallel BC$ (Th. 2.40). But $G$ is the middle point of $AF$, $\therefore E$ is the middle point of $AD$ in the $\triangle AFD$ (Th. 1.26).

(ii) Let $AD$ joining the vertices of the $\triangle$s $ABC$, $DBC$ be bisected by $BC$ in $E$.

**Construction:** Bisect $BC$ in $G$. Join $AG$ and produce it to $F$ so that $GF = AG$. Join $BF$, $FC$, $FD$.

**Proof:** $\therefore AG = GF$ and $BG = GC$, $\therefore ABFC$ is a $\square$ (Th. 1.23). $\therefore \triangle FBC = \triangle ABC$. Again, since $AE = ED$ (hypothesis) and $AG = GF$, $\therefore FD$ is $\parallel GE$ (Th. 1.26, Cor. 1). $\therefore \triangle BDC = \triangle FBC = \triangle ABC$. Hence, in a similar way, it can be shown that:

1. In a triangle $ABC$, the median from $A$ bisects all straight lines parallel to $BC$ and terminated by $AB$ and $AC$ or by these produced.
2. If the base of a triangle be divided into any number of equal parts by straight lines drawn from the vertex, any straight line parallel to the base, which is terminated by the sides or sides produced, will be divided by these straight lines into the same number of equal parts.
2.2. Two triangles are on the same base and between the same parallels. Prove that the sides or sides produced intercept equal segments on any straight line parallel to the base. Let $ABC, DBC$ be the triangles. Draw $GEHF$ parallel to $BC$ meeting $AB, AC, DB, DC$ produced in $E, F, G, H$ respectively. $EF$ will be equal to $GH$.

**Construction:** Through $F$ draw $FLK \parallel AB$; through $G$ draw $GNM \parallel CD$. Produce $BC$ to $L, N$ and $AD$ to $M, K$. Join $BF, GC$ (Fig. 35).

![Figure 35](image)

**Proof:** $\triangle ABC = \triangle DBC$ (Th. 2.39). Also, $\triangle BCF = \triangle BGC$. \therefore by adding, $\triangle ABF = \triangle DGC$. Since $\triangle ABF = \frac{1}{2} \square AL$ (Th. 2.37) and $\triangle DGC = \frac{1}{2} \square CM$, \therefore $\square AL = \square CM$ and since they are between the same \parallel $NL, MK$, \therefore they are on equal bases (converse, Th. 2.35). \therefore $BL = CN$. But $EF = BL$ and $GH = CN$. \therefore $EF = GH$. Similarly, it can be proved that if two triangles are on equal bases and between the same parallels, the sides or sides produced intercept equal segments on any straight line parallel to the base.

2.3. Bisect a triangle by a straight line drawn from a given point on one of its sides.

Let $ABC$ be the given \triangle, $P$ the given point on $AC$.

**Construction:** Bisect $AC$ in $D$. Join $PB$ and draw $DE \parallel PB$. Join $PE, BD$. $PE$ will be the required line (Fig. 36).

![Figure 36](image)
Proof: \( \because AD = DC, \therefore \triangle ADB = \triangle DBC \) (Th. 2.39). \( \therefore \triangle ADB \) is \( \frac{1}{3} \triangle ABC. \) \( \therefore DE \parallel PB, \therefore \triangle PEB = \triangle PDB. \) Adding \( \triangle PAB \) to each, \( \therefore \) fig. \( \triangle PABE = \triangle ADB = \frac{1}{3} \triangle ABC. \) Hence \( PE \) bisects \( \triangle ABC. \)

2.4. Trisect a given quadrilateral by means of two straight lines drawn from (i) one of its vertices; (ii) a given point on one of its sides.

Construction: (i) Let \( ABCD \) be the given quadrilateral. Convert \( \triangle ABCD \) into an equal \( \triangle ADAE \), through the required vertex \( D \), by drawing \( CE \parallel DB \) and joining \( DE \). Trisect \( AE \) in \( F, G \). Draw \( FP, GQ \parallel DB \). Hence \( DP, DQ \) are the two required lines (Fig. 37).

![Figure 37](image)

Proof: \( \because AF = \frac{1}{3} AE, \therefore \triangle DAF = \frac{1}{3} \triangle DAE. \therefore \triangle DBF = \triangle DBP \) (\( FP \parallel DB \)), \( \therefore \triangle DAF = \triangle DABP = \frac{1}{3} \triangle DAE = \frac{1}{3} \triangle ABCD. \)

Similarly, \( \triangle DPQ, DQC \) can be easily shown to be each \( \frac{1}{3} \triangle ABCD. \)

Construction: (ii) Convert \( \triangle \) into an equal \( \triangle D'EF \) through a given point \( D' \) on \( DC \). Trisect \( EF \) in \( P, G \). Draw \( GQ \parallel D'B \). Hence \( D'P, D'Q \) are the two lines (Fig. 38).

![Figure 38](image)

2.5. \( ABCD \) is a parallelogram. \( P, Q \) are taken on \( AB, DP \) so that \( AP = \frac{1}{3} AB, DQ = \frac{1}{3} DP \). Find the area of \( \triangle QBC \) in terms of \( \square ABCD. \)

Construction: Draw \( EQF \parallel AD \) and join \( AQ, BD \) (Fig. 39).
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FIGURE 39

Proof: \( \because DQ = \frac{1}{3} DP, \therefore \triangle ADQ = \frac{1}{3} \triangle ADP \) (Th. 2.39, Cor.). Again, \( AP = \frac{1}{3} AB \). \( \therefore \triangle ADP = \frac{1}{3} \triangle ADB \). Since \( \triangle ADB = \frac{1}{6} \square ABCD \), \( \therefore \triangle ADQ = \frac{1}{3} \times \frac{1}{3} \times \frac{1}{3} \square ABCD = \frac{1}{18} \square ABCD \). But \( \square AFED = 2 \triangle ADQ \). \( \therefore \square AFED = \frac{1}{9} \square ABCD \). Hence \( \square FBCE = \frac{5}{9} \square ABCD \).

Since \( EF \parallel AD \) and \( BC \) and \( Q \) lies on \( EF \), \( \therefore \triangle QBC = \frac{1}{3} \square FBCE \) (Th. 2.37). Therefore \( \triangle QBC = \frac{1}{3} \times \frac{5}{9} = \frac{5}{27} \square ABCD \).

2.6. A square is drawn inside a triangle so that one of its sides coincides with the base of the triangle and the other two corners lie on the other two sides of the triangle. Show that twice the area of the triangle is equal to the rectangle contained by one side of the square and the sum of base of the triangle and the altitude on this base.

CONSTRUCTION: Let \( DEFG \) be the inscribed square. Draw \( AN \perp BC \) to cut \( GD \) in \( M \). Then join \( CD \) (Fig. 40).

FIGURE 40

Proof: \( 2 \triangle DBC = CB \times DE \). Also, \( 2 \triangle GDC = GD \times GF \) (since \( GF \perp \) from \( C \) to \( GD \)) and \( 2 \triangle ADG = GD \times AM \). \( \therefore DE = GD \), adding yields \( 2(\triangle DBC + \triangle GDC + \triangle ADG) = DE (CB + GF + AM) = DE (CB + AN) \) or \( 2 \triangle ABC = DE (CB + AN) \).

2.7. If \( E \) is the point of intersection of the diagonals in the parallelogram
ABCD and P is any point in the triangle ABE, prove that \( \triangle PDC = \triangle ABP + \triangle PBD + \triangle PAC \).

**Construction:** Draw QPR \( \parallel \) AB to meet AD, BC in Q, R respectively (Fig. 41).

\[ \text{Proof: } \triangle ABD = \triangle ABP + \triangle APD + \triangle PBD = \frac{1}{2} \square ABCD. \]

Also, \( \triangle ABC = \triangle ABP + \triangle BPC + \triangle PAC = \frac{1}{2} \square ABCD. \)

By adding, \( 2 \triangle ABP + \triangle APD + \triangle BPC + \triangle PBD + \triangle PAC = \square ABCD \) (1). But \( \triangle ABP = \frac{1}{2} \square AQRB \) and \( \triangle PDC = \frac{1}{2} \square DQRC \). \( \therefore \triangle ABP + \triangle PDC = \frac{1}{2} \square ABCD \) (2). Therefore, the remainder \( \triangle APD + \triangle BPC = \frac{1}{2} \square ABCD \) (3). Subtracting (3) from (1) gives \( 2 \triangle ABP + \triangle PBD + \triangle PAC = \frac{1}{2} \square ABCD \) (4). Now, by equating (2) and (4), \( \triangle PDC = \triangle ABP + \triangle PBD + \triangle PAC \).

2.8. ABC is a right-angled triangle at A. Two squares ABFG and ACKL are described on AB and AC outside the triangle. If BK, CF cut AC, AB in M, N, show that AM = AN.

\[ \text{Analysis: } \text{Join } MN, FM, KN \text{ (Fig. 42). Assume } AM \text{ is equal to } AN; \therefore \angle AMN = \angle ANM = \frac{1}{2} \text{ a right angle } = \angle CAK. \therefore MN \parallel FK \text{ (FK is a straight line since } \angle BAC \text{ is right angle). } \therefore \triangle FMN \]
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\( \triangle KMN \) (Th. 2.39). By adding \( \triangle OMN \), \( \triangle FOM = \triangle KON \). But this is the case, for \( \triangle FMB = \triangle FCB \), since \( FB \parallel AC \) (Th. 2.39). Take away \( \triangle FOB \); \( \therefore \) \( \triangle FMO = \triangle OBC \). Similarly, \( \triangle KNO = \triangle OBC \); \( \therefore \) \( \triangle FMO = \triangle KNO \). Hence,

SYNTHESIS: \( \therefore \) \( \triangle FMB = \triangle FCB \), take \( \triangle FOB \) from each; \( \therefore \) \( \triangle FMO = \triangle OBC \). Similarly, \( \triangle KNO = \triangle OBC \), \( \therefore \) \( \triangle FMO = \triangle KNO \). \( \therefore \) \( \triangle AMN = \triangle ANM \). \( \therefore \) \( AM = AN \).

2.9. Construct a parallelogram that will be equal in area and perimeter to a given triangle.

FIGURE 43

ANALYSIS: Let \( ABC \) be the given triangle. Assume that \( BDEF \) is the required \( \square \) (Fig. 43). Since \( \square BDEF \) will be equal in area to the given \( \triangle ABC \), then it should lie on half the base \( BC \) and between \( BC \) and the parallel to it through \( A \). Hence \( D \) is the mid-point of \( BC \).

\( CB = 2 \ DB = DB + EF \). Since also \( \square BDEF \) will be equal in perimeter to \( \triangle ABC \), \( \therefore \) \( AB + AC = 2 \ DE = DE + FB \); i.e., \( DE = \frac{1}{2} \ (AB + AC) \). This is true if \( DE \) will be equal to the line joining the midpoint of \( BC \) to the foot of the perpendicular from \( C \) or \( B \) on the external bisector of \( \angle A \) (see Problem 1.10). Thus,

SYNTHESIS: Bisect \( BC \) in \( D \) and draw \( CH \perp \) the external bisector of \( \angle A \). Join \( DH \) and draw \( AF \parallel BC \). With \( D \) as center and \( DH \) as radius, take \( DE = DH \). Then \( BDEF \) is the required \( \square \).

Proof: Produce \( CH \) to meet \( BA \) produced in \( G \). Join \( AD \). \( \triangle s \) \( ACH, AGH \) are congruent (Th. 1.11). \( \therefore \) \( CH = HG \) and \( AC = AG \). In the \( \triangle BCG \), \( DH = \frac{1}{2} \ BG = \frac{1}{2} \ (AB + AC) = DE \). \( \therefore \) \( AB + AC = DE + BF \). Since \( BC = BD + EF \), \( \therefore \) perimeter of \( \square BDEF \) = perimeter of \( \triangle ABC \). \( \therefore \) \( AF \parallel BC \), \( \therefore \) \( \square BDEF = \triangle ABC = 2 \ \triangle ABD \).

2.10. \( ABC \) is a triangle, and \( D, E \) are any two points on \( AB, AC \) respectively. \( AB, AC \) are produced to \( G, H \), so that \( BG = AD \) and \( CH = AE \). If \( BH \),
CG intersect in L and DE, GH are joined, show that \( \triangle LGH = \triangle ADE + \triangle LBC \).

**Construction:** Join CD, HD (Fig. 44).

**Figure 44**

Proof: \( \triangle ADE = \triangle CDH \) (have equal bases and altitudes, Th. 2.39). But, \( \triangle ADH = \triangle ADC + \triangle CDH = \triangle ADC + \triangle ADE \).

Since also \( \triangle ADH = \triangle HBG \) and \( \triangle ADC = \triangle CBG \) (Th. 2.39), \( \therefore \triangle HBG = \triangle CBG + \triangle ADE \).

By subtracting the common \( \triangle LBG \) from \( \triangle s HBG, CBG, \therefore \triangle LGH = \triangle ADE + \triangle LBC \).

2.11. \( ABCD \) is a parallelogram. From A and C, two parallel lines AE, CF are drawn to meet BC, AD in E, F. If a line is drawn from E parallel to AC to meet AB in P, show that PF \( \parallel \) BD.

**Construction:** Join PC, PD, BF (Fig. 45).

**Figure 45**

Proof: \( \triangle AFB = \triangle AFC \) (on same base and between same parallels, Th. 2.39). \( \therefore AE \parallel FC, \therefore AECF \) is a ||. \( \therefore \triangle AFC = \triangle ACE \) (Th. 2.42). \( \therefore \triangle AFB = \triangle ACE \).

Since \( \triangle ACE = \triangle APC \) (because PE \( \parallel \) AC) and \( \triangle APC = \triangle APD \) (since AP \( \parallel \) DC), \( \therefore \triangle AFB = \triangle APD \).

By subtracting common \( \triangle AFP \) from both, \( \therefore \triangle FPB = \triangle FPD \).

Since these two \( \triangle s \) are on the same base FP, \( \therefore FP \parallel BD \) (Th. 2.40).

2.12. A point D is taken inside a triangle ABC. On the sides of the triangle ABC, the rectangles BCEF, CAGH, ABKL are drawn outside the triangle such that the area of each rectangle = twice the area of \( \triangle ABC \). Prove that the sum of the areas of \( \triangle s DEF, DGH, DKL \) equals four times \( \triangle ABC \).
CONSTRUCTION: Draw $DR \perp EF$. Join $AD, BD, CD$ (Fig. 46).

**Figure 46**

**Proof:** $\triangle DBF = \frac{1}{2} \square BR$ (Th. 2.38). Also, $\triangle DCE = \frac{1}{2} \square CR$. By adding, $\triangle DBF + \triangle DCE = \frac{1}{2} \square BCEF$. Since $\triangle DEF = \text{fig. DBFEC} - (\triangle DBF + \triangle DCE)$, $\therefore \triangle DEF = \triangle DBC + \square BCEF - \frac{1}{2} \square BCEF = \triangle DBC + \frac{1}{2} \square BCEF$. Similarly, $\triangle DGH = \triangle DCA + \frac{1}{2} \square CAGH$ and $\triangle DKL = \triangle DAB + \frac{1}{2} \square ABKL$. Since these rectangles are each $= 2 \triangle ABC$ (hypothesis), adding gives $\triangle DEF + \triangle DGH + \triangle DKL = \triangle ABC + \frac{1}{2} (\square BCEF + \square CAGH + \square ABKL) = 4 \triangle ABC$.

2.13. $ABC$ is a triangle having the angle $C$ a right angle. Equilateral triangles $ADB, AEC$ are described externally to the triangle $ABC$ and $CD$ is drawn. Show that $\triangle ACD = \triangle AEC + \frac{1}{2} \triangle ABC$.

**Construction:** Bisect $AB$ in $F$, then join $EF, CF, EB$. Let $EF$ cut $AC$ in $G$ (Fig. 47).

**Figure 47**
Proof: ∠BAD = ∠CAE = 60° (angles of equilateral Δs), then adding ∠CAB to each gives ∠CAD = ∠BAE.

Now, Δs ACD, AEB are congruent (Th. 1.10) ∴ They are equal in area; i.e., ΔACD = ΔAEB. But ΔABC is right-angled at C, and F is the mid-point of AB. ∴ CF = AF (Th. 1.27). ∴ Δs ECF, EAF are congruent (Th. 1.14). ∴ ∠CEF = ∠AEF. Consequently, Δs GEC, GEA are congruent (Th. 1.10). ∴ ∠CGE = ∠AGE = right angle. Hence EF ⊥ AC or || CB. ∴ ΔAEB = ΔAEF + ΔFEB = ΔAEF + ΔFEC (since EF || CB), ∴ ΔAEB = ΔAFCE = ΔAEC + ΔACF = ΔAEC + ½ ΔABC. Since ΔACD = ΔAEB (as shown above), ∴ ΔACD = ΔAEC + ½ ΔABC.

2.14. ABC is a triangle right-angled at A. AD is drawn perpendicular to BC. If two squares BE, CF are described on AB, AC each on the same side of its base as the triangle ABC, show that the triangle ABC is equal to the triangle DEF together with the square on AD.

Construction: Draw FG, EH ⊥ AD. Join BE, CF (Fig. 48).

[Diagram Figure 48]

Proof: FG || BC, ∴ ∠AFG = ∠ABC. But ∠ABC = ∠CAD (both are complementary to ∠BAD). ∴ AF = AC (in the square CF), ∴ Δs AGF, CDA are congruent (Th. 1.11). ∴ FG = AD. Similarly, Δs ADB, EHA are congruent. ∴ AD = EH = FG. Now, ΔAEF = ΔDEF + ΔADF + ΔADE = ΔDEF + ½ AD·FG + ½ AD·EH = ΔDEF + AD².

2.15. ABCD is a quadrilateral and G, H are the middle points of AC, DB. If CB, DA are produced to meet in E, show that ΔEGH = ΔABCD.

Construction: Join GB, GD, HA, HC (Fig. 49).

Proof: G is the mid-point of AC, ∴ ΔEGC = ½ ΔAEC (Th. 2.39). Also, ΔDGC = ½ ΔDCA. ∴ ΔEGC + ΔDGC = ½ (ΔACE + ΔDCA) = ½ ΔEDC (1). Similarly, ΔBGC + ΔDGC =
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\[ \frac{1}{2} \triangle ABCD \] 

(2). Hence, subtracting (2) from (1), \( \triangle EBG = \frac{1}{2} \triangle ABE \). Similarly, \( \triangle EHD + \triangle HDC = \frac{1}{2} \triangle EDC \) and \( \triangle AHD + \triangle HDC = \frac{1}{2} \triangle ABCD \). By subtracting, \( \triangle AHE = \frac{1}{2} \triangle ABE \).

Again, \( \triangle EHC = \frac{1}{2} \triangle EDC \). But \( \triangle AHE = \frac{1}{2} \triangle ABE = \triangle EBG \).

\[ \therefore \triangle EGH + \triangle BGC + \triangle HGC = \frac{1}{2} \triangle ABCD \] 

(3) and \( \triangle BGC + \triangle HGC = \frac{1}{2} \triangle ABCH = \frac{1}{4} \triangle ABCD \) 

(4). Hence from (3) and (4), \( \triangle EGH = \frac{1}{4} \triangle ABCD \).

2.16. The middle points of the three diagonals of a complete quadrilateral are collinear.

CONSTRUCTION: Let \( ABCD \) be the complete quadrilateral and \( G, H, J \) the middle points of its diagonals \( AC, BD, EF \). Join \( EG, EH, FG, FH \). Draw \( EL, FK \perp GH, GH \) produced. Assuming that \( GH \) produced cuts \( EF \) in \( J \), the problem is then reduced to prove that \( J \) is the mid-point of \( EF \) (Fig. 50).

\[\text{Figure 49}\]

\[\text{Figure 50}\]
**Proof:** According to Problem 2.15, \( \triangle EGH = \frac{1}{4} \triangle ABCD \). Similarly, \( \triangle FGH = \frac{1}{4} \triangle ABCD \). \( \therefore \triangle EGH = \triangle FGH \). Since these \( \triangle \)s are on the same base \( GH \), \( \therefore \) their altitudes on \( GH \) should be equal; i.e., \( EL = FK \) (Th. 2.41). Hence \( \triangle \)s \( ELJ, FKJ \) are congruent (Th. 1.11). \( \therefore \) \( EJ = JF \) or \( J \) is the mid-point of \( EF \). Therefore, \( G, H, J \) are collinear.

2.17. \( ABCD \) is a square and \( E \) is the intersection point of the diagonals. If \( N \) is any point on \( AE \), show that (i) \( AB^2 - BN^2 = AN \cdot NC \); (ii) \( AN^2 + NC^2 = 2 BN^2 \) (Fig. 51).

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure51.png}
\caption{Figure 51}
\end{figure}

**Proof:** (i) The diagonals of a square are \( \perp \) and bisect one another. \( \therefore \) \( AE = EC \) and \( \perp BE \). \( AN \cdot NC = AN(NE + EC) = AN(AN + 2 NE) = AN^2 + 2 AN \cdot NE \). In \( \triangle ABN \), \( AB^2 = BN^2 + AN^2 + 2 AN \cdot NE \) (Th. 2.53). Hence \( AB^2 - BN^2 = AN \cdot NC \).

**Corollary 2.17.1.** This is also true for any-angled isosceles triangle \( ABC \) having \( \angle B \) other than a right angle.

(ii) \( AN^2 = (AE - NE)^2 \) and \( NC^2 = (CE + NE)^2 = (AE + NE)^2 \). Adding gives \( AN^2 + NC^2 = 2 (AE^2 + NE^2) = 2 (BE^2 + NE^2) = 2 BN^2 \).

2.18. In any trapezoid, the sum of the squares on the diagonals is equal to the sum of the squares on the non-parallel sides plus twice the rectangle contained by the parallel sides.

**Construction:** Draw the \( \perp \)s \( CF, DG \) on \( AB \) produced (Fig. 52).

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure52.png}
\caption{Figure 52}
\end{figure}
Proof: In $\triangle ABD$, $ABC$, $BD^2 = AD^2 + AB^2 + 2 AB \cdot AG$, $AC^2 = BC^2 + AB^2 + 2 AB \cdot BF$ (Th. 2.53). Adding yields $BD^2 + AC^2 = AD^2 + BC^2 + 2 AB (AB + AG + BF)$ or $BD^2 + AC^2 = AD^2 + BC^2 + 2 AB \cdot CD$.

2.19. $ABC$ is a right-angled triangle at $A$. Show that, if $AD$ is the perpendicular from $A$ to the hypotenuse and denoting $AB$, $AC$, $BC$, $AD$ by $c$, $b$, $a$, $e$, then $1/e^2 = (1/c^2) + (1/b^2)$ (Fig. 53).

![Figure 53](image_url)

Proof: $ae = be = twice \triangle ABC$. $\therefore e = bc/a$ or $e^2 = b^2c^2/a^2$. $\therefore 1/e^2 = a^2/b^2c^2$. $\therefore a^2 = b^2 + c^2$ (Th. 2.51), therefore, $1/e^2 = (b^2 + c^2)/b^2c^2 = (1/c^2) + (1/b^2)$.

2.20. Any point $P$ is taken inside or outside triangle $ABC$. From $P$ perpendiculars $PE$, $PD$, $PF$ are drawn to the sides $BC$, $CA$, $AB$ respectively. Show that $AF^2 + BE^2 + CD^2 = BF^2 + CE^2 + AD^2$. Enunciate and prove the converse of this theorem.

Construction: Join $PA$, $PB$, $PC$ (Fig. 54).

![Figure 54](image_url)

Proof: $AF^2 - BF^2 = AP^2 - BP^2$ (Th. 2.51 Cor. 1). Also, $BE^2 - CE^2 = BP^2 - CP^2$ and $CD^2 - AD^2 = CP^2 - AP^2$. Hence adding yields $(AF^2 - BF^2) + (BE^2 - CE^2) + (CD^2 - AD^2) = 0$ or $AF^2 + BE^2 + CD^2 = BF^2 + CE^2 + AD^2$. Now, the converse will be: If the above expression in a triangle $ABC$ is true, then the perpendiculars through $D$, $E$, $F$ are concurrent. This is evident, since if the $\perp$s from $E$, $F$ meet in $P$, then $PD$ should be $\perp AC$. From the
above \((AF^2 - BF^2) + (BE^2 - CE^2) = (AP^2 - BP^2) + (BP^2 - CP^2)\). Hence the remainders are equal or 
\((CD^2 - AD^2) = (CP^2 - AP^2)\). According to (Th. 2.51, Cor. 2), this is true when \(DP\) is \(\perp AC\). Therefore, the \(\perp\)'s from such points \(D, E, F\) are concurrent.

2.21. \(ABC\) is an isosceles triangle having \(AB = AC\). Find (i) point \(D\) on \(BC\) so that, if \(DE\) is drawn \(\perp BC\) to meet \(AB\) in \(E\), then \(AD^2 + DE^2 = AB^2\); (ii) point \(F\) on \(CB\) produced so that, if \(FG\) is \(\perp BC\) to meet \(AB\) produced in \(G\), then \(AF^2 - FG^2 = AB^2\).

**Figure 55**

\[\text{ANALYSIS:} \ (i) \ \text{Suppose } D \text{ is the required point (Fig. 55). Then, } AD^2 + DE^2 = AB^2. \text{ According to Problem 2.17(i), } AB^2 = AD^2 + BD \cdot DC. \text{ Hence } DE^2 = BD \cdot DC. \text{ But this is only true if } \angle CEB = \text{ right angle. Therefore,} \]

\[\text{SYNTHESIS: Draw } CE \perp AB \text{ and from } E \text{ draw } ED \perp BC. \text{ Then } D \text{ is the required point.} \]

**Figure 56**

\[\text{ANALYSIS:} \ (ii) \ \text{Suppose } F \text{ is the required point (Fig. 56). Then } AF^2 - FG^2 = AB^2. \text{ In the } \triangle ABF, AF^2 = AB^2 + BF^2 + 2 BF. \]
BM (Th. 2.53) or $AF^2 - BF \cdot FC = AB^2$. Hence $FG^2 = BF \cdot FC$, which is true if $BF$ is made equal to $FH$ and $\triangle CGH$ right-angled at $G$. Thus $\triangle BGH$ is isosceles and $\angle BGF = \angle FGH$. But since $\angle BGF = \angle BED = \angle ECD$ and $\angle FGH = \angle FCG$, $\therefore \angle ECD$ or $JCF = \angle FCG$. Hence $FG = FJ$. $\therefore EF = FJ$ (since $JEG$ is a right angle). $\therefore \angle FEJ = \angle FJE = \angle DEC$ (since $FJ \parallel DE$). Therefore $FE$, $DE$ are equally inclined to $CJ$, and thus,

**Synthesis:** Draw $CE \perp AB$ and produce it to $J$; then draw $EF$ making with $EJ$ an $\angle FEJ = \angle DEC$ and meeting $CB$ produced in $F$ which is the required point.

2.22. $ABC$ is a triangle and three squares $BCDE$, $ACFG$, $ABHK$ are constructed on its sides outside the triangle. Show that (i) $AB^2 + AD^2 = AC^2 + AE^2$; (ii) $GK^2 + CB^2 = 2 (AB^2 + AC^2)$; (iii) $AD^2 + BG^2 + CH^2 = AE^2 + BF^2 + CK^2$; (iv) $GK^2 + HE^2 + DF^2 = 3 (AB^2 + BC^2 + AC^2)$; (v) area of hexagon $GKHEDF = 4 \triangle ABC + (AB^2 + BC^2 + AC^2)$.

**Construction:** Draw $AMN \perp BC$, $DE$ and $AP \perp GK$. Produce $PA$ and draw $CQ$, $BR \perp$ s to it (Fig. 57).

**Figure 57**

Proof: (i) $AB^2 + AD^2 = AM^2 + BM^2 + AN^2 + DN^2$ (Th. 2.51). Also, $AC^2 + AE^2 = AM^2 + CM^2 + AN^2 + NE^2$. Since $BM = NE$ and $DN = CM$, $\therefore AB^2 + AD^2 = AC + AE^2$.

(ii) $\triangle BAG$, $KAC$ are congruent. $\therefore \angle ABG = \angle AKC$. $\therefore \angle AKC + \angle AJK = \text{right angle}$, $\therefore \angle AJK \text{ or } \angle BJL + \angle ABG = \text{right angle}$. Hence, $BG \perp CK$. Now, in the quadrilateral $BCGK$, the
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diagonals are \( \perp \) to one another. Hence \( GK^2 + CB^2 = BK^2 + CG^2 = 2 (AB^2 + AC^2) \).

(iii) According to (i), \( AB^2 + AD^2 = AC^2 + AE^2 \). Similarly, \( BC^2 + BG^2 = AB^2 + BF^2 \) and \( AC^2 + CH^2 = BC^2 + CK^2 \). Adding gives \( AD^2 + BG^2 + CH^2 = AE^2 + BF^2 + CK^2 \). Alternatively, \( BG = CK \) from congruence of \( \triangle BAG, KAC \) above and similarly it is easily shown that \( AD = BF \) and \( CH = AE \). Hence (iii) is true again.

(iv) As in (ii), \( HE^2 + AC^2 = 2 (BC^2 + AB^2) \) and \( DF^2 + AB^2 = 2 (AC^2 + BC^2) \). Therefore, by adding, \( GK^2 + HE^2 + DF^2 = 3 (AB^2 + BC^2 + AC^2) \).

(v) According to Problem 1.31(ii), \( \triangle APG, CQA \) and \( \triangle APK, BRA \) are congruent. \( \because \) \( AQR \) is a median in \( \triangle ABC \). \( \therefore \) \( \triangle CQA, \triangle BRA = \triangle ABC = \triangle APG + \triangle APK = \triangle AGK \). Similarly, \( \triangle ABC = \triangle BHE = \triangle CDF \). Therefore, hexagon \( GKHEDF = 4 \triangle ABC + (AB^2 + BC^2 + AC^2) \).

2.23. ABCD is a quadrilateral. E, F are the middle points of the diagonals AC, BD. If G is the mid-point of EF and P is any point outside the quadrilateral, show that \( AP^2 + BP^2 + CP^2 + DP^2 = AG^2 + BG^2 + CG^2 + DG^2 + 4PG^2 \).

CONSTRUCTION: Join PE, PF (Fig. 58).

![Figure 58](image)

**Proof:** In the \( \triangle APC, AP^2 + PC^2 = 2PE^2 + 2AE^2 \) (Th. 2.55). Similarly, in the \( \triangle PBD, PB^2 + PD^2 = 2PF^2 + 2BF^2 \). Hence, by adding, \( AP^2 + BP^2 + CP^2 + DP^2 = 2 (PE^2 + PF^2 + AE^2 + BF^2) \) (1). In the \( \triangle PEF, 2 (PE^2 + PF^2) = 4 (PG^2 + GF^2) = 4 PG^2 + 2GE^2 + 2GF^2 \) (2). In \( \triangle AGC, BGD, AG^2 + BG^2 + CG^2 + DG^2 = 2 (AE^2 + BF^2) + 2 (GE^2 + GF^2) \) (3). Hence, from these equations, \( AP^2 + BP^2 + CP^2 + DP^2 = AG^2 + BG^2 + CG^2 + DG^2 + 4PG^2 \).

2.24. Divide a straight line into two parts, so that the rectangle contained by the whole and one part may be equal to the square on the other part.
CONSTRUCTION: (i) Let $AB$ be the given straight line. It is required
to divide it internally in $K$, so that $AK^2 = AB \cdot BK$. On $AB$
describe square $ACDB$. Bisect $AC$ in $E$. Join $EB$ and produce $CA$
to $G$ making $EG = EB$. On $AG$ describe square $AGHK$, then $K$
is the required point. Produce $HK$ to $L$ [Fig. 59 (i)].

Proof: \( \because \) $CA$ is bisected in $E$ and produced to $G$, \( \therefore \) $EG^2 = CG \cdot GA + EA^2 = EB^2 = EA^2 + AB^2$, since $EG = EB$. Hence $CG \cdot GA = AB^2 = \text{fig. AD}$. But fig. $GL = CG \cdot GA$. \( \because \) $GA = GH$, \( \therefore \) fig. $GL = \text{fig. AD}$. Take from each fig. $AL$. \( \therefore \) fig. $GK = \text{fig. KD}$ or $AK^2 = AB \cdot BK$.

CONSTRUCTION: (ii) To divide the line externally, describe square $ABDC$, bisect $AC$ in $E$. Join $EB$, and produce $AC$ through $C$ to $G'$, making $EG' = EB$. On $AG'$, on the side away from $AD$, describe square $AG'H'K'$. Produce $DC$ to meet $H'K'$ in $L'$ [Fig. 59(ii)]. Then it can be proved that $C'G' \cdot G'A = AB^2$, as before. Hence fig. $G'L' = \text{fig. AD}$. Add to each fig. $AL'$. \( \therefore \) fig. $G'K' = \text{fig. KD}$ and fig. $K'D = AB \cdot BK'$. \( \because \) $AB = BD$, \( \therefore \) $AB \cdot BK' = AK'^2$.

Algebraic equivalent: If in the previous problem $AB$ and $AK$ contain $a$ and $x$ units of length respectively, then $a(a - x) = x^2$; \( \because \) $x^2 + ax - a^2 = 0$. Hence $AK$ and $AK'$ correspond to the roots of $x$ in this quadratic equation.

2.25. $ABC$ and $DEF$ are two triangles, $DEF$ being the greater, which are
so located that each pair of corresponding sides are parallel. From $A$, $C$ two
perpendiculars $AG$, $CH$ are drawn to $DF$, from $B$, $C$ another two perpen-
diculars $BK$, $CL$ are drawn to $EF$, and from $A$, $B$ a third pair of perpendicu-
lars $AM$, $BN$ are drawn to $DE$. Show that $AK^2 + BH^2 + CM^2 = AL^2 + BG^2 + CN^2$.

CONSTRUCTION: From $A$, $B$, $C$ draw $AP'P$, $QBQ'$, $CR'R$ \( \perp \) to each
pair of opposite parallel sides of the \( \triangle $s (Fig. 60).

Proof: $AK^2 + BH^2 + CM^2 = AP^2 + PK^2 + BQ^2 + QH^2 + CR^2$.
+ RM^2 \quad (\text{Th. 2.51}) \quad (1). \ \text{Again, } AL^2 + BG^2 + CN^2 = AP^2 + PL^2 + BQ^2 + QG^2 + CR^2 + RN^2 \quad (2). \ \text{But, since the sides are } \parallel, \ \text{the } \perp \text{s from } A, B, C \text{ on the sides of } \triangle DEF \text{ are the altitudes of } \triangle ABC. \ \text{Hence they meet at one point } O \ (\text{Th. 1.33}). \ \therefore \text{According to Problem 2.20, } BP'^2 + CQ'^2 + AR'^2 = CP'^2 + AQ'^2 + BR'^2 \text{ or } PK^2 + QH^2 + RM^2 = PL^2 + QG^2 + RN^2. \ \text{Then subtracting from (1) and (2), } AK^2 + BH^2 + CM^2 = AL^2 + BG^2 + CN^2.

2.26. \text{If in a quadrilateral the sum of the squares on one pair of opposite sides is equal to the sum of the squares on the other pair, the diagonals will be perpendicular to one another and the lines joining the middle points of opposite sides are equal.}

\text{CONSTRUCTION: Let } ABCD \text{ be a quadrilateral in which } AB^2 + CD^2 = BC^2 + AD^2, \text{ and let } E, F, G, H \text{ be the middle points of } AB, BC, CD, DA \text{ (Fig. 61). Let } AC, BD \text{ meet in } M \text{ and draw } AR, CP \perp BD. \ \text{Join } EF, FG, GH, HE. \ (P, R \text{ must fall on opposite sides of } AC.)
Proof: (i) In the \( \triangle ABM \), \( AB^2 = AM^2 + BM^2 + 2BM \cdot RM \) (Th. 2.53), and in the \( \triangle CDM \), \( CD^2 = CM^2 + DM^2 + 2DM \cdot PM \). Hence, by adding, \( AB^2 + CD^2 = AM^2 + BM^2 + CM^2 + DM^2 + 2BM \cdot RM + 2DM \cdot PM \). Similarly, \( AD^2 + BC^2 = AM^2 + BM^2 + CM^2 + DM^2 - 2BM \cdot PM - 2DM \cdot RM \) (Th. 2.54). \( \therefore AB^2 + CD^2 = AD^2 + BC^2 \). \( \therefore 2BM \cdot RM + 2DM \cdot PM = -2BM \cdot PM - 2DM \cdot RM \). \( \therefore (PM + RM)(BM + DM) = 0 \) or \( PR \cdot BD = 0 \), which is impossible unless \( PR = 0 \). Hence \( P, R \) coincide with \( M \), thus making \( AR, CP \) one line with \( AC \perp DB \).

(ii) Since the lines joining the middle points \( EF, GH \) are \( \parallel AC \) and \( HE, FG \) are \( \parallel BD \) and since \( AC \perp DB \), \( EFGH \) is a rectangle and thus its diagonals \( EG, FH \) are equal.

2.27. \( ABC \) is an isosceles triangle in which \( AB = AC \). From \( C \), a perpendicular \( CD \) is drawn to \( AB \) (Fig. 62). Denoting \( AB, AC, BC \) by \( b, b, a \), show that (i) \( BC^2 = 2 AB \cdot BD \); (ii) \( CD = \frac{(a/2b)\sqrt{4b^2 - a^2}}{4} \).

\[ \text{Figure 62} \]

Proof: (i) \( BC^2 = AB^2 + AC^2 - 2AB \cdot AD \) (Th. 2.54) = \( 2AB^2 - 2AB \cdot AD = 2AB(AB - AD) = 2AB \cdot BD \).

(ii) \( CD^2 = a^2 - DB^2 \). \( \therefore a^2 = 2b \cdot BD \). \( \therefore BD = a^2/2b \). \( \therefore CD = \sqrt{(4a^2b^2 - a^4)/4b^2} = (a/2b)\sqrt{4b^2 - a^2} \).

2.28. \( ABC \) is a triangle and \( O \) any point. The parallelograms \( AOBC', BOCA', COAB' \) are completed. Show that the lines \( AA', BB', CC' \) are concurrent and that the sum of the squares on their lengths is equal to the sum of the squares on the sides of the triangle \( ABC \) and on the distances of \( O \) from its vertices.

Construction: Join \( A'C', B'C', A'B' \) (Fig. 63).

Proof: Since \( AB', BA' \) are equal and \( \parallel \) to \( CO \), \( \therefore ABA'B' \) is a \( \square \). Also, \( BCB'C', ACA'C' \) are \( \square \)'s. \( \therefore AA', BB', CC' \) are diagonals in these \( \square \)'s taken in pairs, each pair bisects one another and consequently they are concurrent at their middle point \( Q \). Now, \( AA'^2 + BB'^2 = 2(AB^2 + A'B'^2) = 2(AB^2 + CO^2) \). Similarly, \( AA'^2 + \)
AREAS, SQUARES, AND RECTANGLES

Figure 63

\[ CC' = 2(AC^2 + A'C^2) = 2(AC^2 + BO^2) \] and \[ BB' + CC' = 2(BC^2 + BC^2) = 2(BC^2 + AO^2). \] Adding yields \[ AA' + BB' + CC' = (AB^2 + AC^2 + BC^2) + (AO^2 + BO^2 + CO^2). \]

Note: This theorem is interesting to consider in three dimensions by taking point \( O \) outside the plane of the \( \triangle ABC \). Then the diagram can be interpreted as a plane diagram of a three-dimensional figure.

2.29. \( ABCD \) is a square. Perpendiculars \( AA', BB', CC', DD' \) are drawn to any straight line outside the square. Prove that \( AA'^2 + CC'^2 - 2 BB' - DD' = BB'^2 + DD'^2 - 2 AA' - CC' = \text{area of } ABCD. \)

Construction: Join the diagonals \( AC, BD \), to intersect in \( O \). Draw \( OO' \) \perp the straight line and the perpendiculars \( AX, BY, DZ \) on \( OO' \) or produced; then join \( AO', CO' \) (Fig. 64).

Proof: \( O \) is the mid-point of \( AC, BD \), \( \therefore \) according to 1.17, \( AA' + CC' = BB' + DD' = 2OO' \). Hence \( (AA' + CC')^2 = (BB' + DD')^2 \), yielding \( AA'^2 + CC'^2 - 2 BB' \cdot DD' = BB'^2 + DD'^2 - \)
2.30. Describe a square equal to a given rectilinear figure.

**Construction:** Let ABCDE be the given figure. Convert this figure into an equal triangle through one of its vertices D, by drawing CF, EG \parallel BD, AD respectively, where F, G are on AB produced. \( \therefore \) \( \triangle DFG = \text{fig. ABCDE (2.4)} \). Bisect FG in K and draw the rectangle FIJK on FK to meet the \( \parallel \) from D to AB in IJ. If FK = FI, then FJ would be the required square. If not, produce KF to N, so that FN = FL. Bisect KN in M, and with M as center and radius MK, describe a semi-circle KLN. Produce IF to L. Then the square on FL will be equal to fig. ABCDE. Join LM (Fig. 65).

![Figure 65](image)

**Proof:** \( \therefore \) M is the mid-point of KN and F is another point on KN, \( \therefore \) \( KF \cdot FN + FM^2 = MN^2 = ML^2 \), \( \therefore \) \( MN = ML = FM^2 + FL^2 \). \( \therefore \) \( FL^2 = FK \cdot FN = FK \cdot FI = \text{rectangle FJ} \). But rectangle FJ = 2 \( \triangle DFK = \triangle DFG = \text{fig. ABCDE} \). Hence FL = \( \text{fig. ABCDE} \).

2.31. ABCD is any quadrilateral. Bisect the sides AB, BC, CD and DA in E, F, G and H respectively. Join EG and FH, which intersect in O. If the diagonals AC, BD are bisected in L, M respectively show that (i) LOM is one straight line and O bisects LM; (ii) \( OA^2 + OB^2 + OC^2 + OD^2 = EG^2 + FH^2 + LM^2 \).
Construction: Join \(EH, EF, LB, LD, LE, GL, EM,\) and \(GM\) (Fig. 66).

\[
\begin{align*}
\text{Proof:} & \quad \text{(i) Since } L, E, G, \text{ and } M \text{ are the mid-points of } AC, AB \text{ and } DC, DB \text{ in } \triangle ABC, DBC, \therefore LE = GM = \frac{1}{2} BC \text{ and } || \text{ to it. Therefore, } LEMG \text{ is a } \square. \therefore LOM \text{ is one straight line and } O \text{ bisects } LM, GE, FH \text{ (see Problem 1.14).} \\
& \quad \text{(ii) In } \triangle ABC, ADC, AB^2 + BC^2 + CD^2 + DA^2 = 2 BL^2 + 2 DL^2 + 4 AL^2. \therefore \text{In } \triangle BLD, 2 BL^2 + 2 DL^2 = 4 LM^2 + 4 BM^2, \text{ hence } AB^2 + BC^2 + CD^2 + DA^2 = AC^2 + BD^2 + 4 LM^2. \\
& \quad \text{Now,} \\
OA^2 + OB^2 &= 2 AE^2 + 2 OE^2, \\
OB^2 + OC^2 &= 2 BF^2 + 2 OF^2, \\
OC^2 + OD^2 &= 2 CG^2 + 2 OG^2, \\
OD^2 + OA^2 &= 2 DH^2 + 2 OH^2. \\
\therefore 4 (OA^2 + OB^2 + OC^2 + OD^2) &= 4 (OB^2 + OF^2 + OG^2 + OH^2) + 4 (AE^2 + BF^2 + CG^2 + DH^2) \\
&= 2 (EG^2 + HF^2) + (AB^2 + BC^2 + CD^2 + DA^2) \\
&= 2 (EG^2 + HF^2) + (AC^2 + BD^2 + 4 LM^2) \\
&= 2 (EG^2 + HF^2) + 4 (EF^2 + EH^2 + LM^2) \\
&= 2 (EG^2 + HF^2) + 8 (OE^2 + OH^2) + 4 LM^2 \\
&= 4 (EG^2 + FH^2 + LM^2). \\
\text{Hence} \\
OA^2 + OB^2 + OC^2 + OD^2 &= EG^2 + FH^2 + LM^2.
\end{align*}
\]

2.32. \(ABC, DEF\) are two triangles so located that the perpendiculars from \(A, B, C\) on \(EF, DF, DE\) respectively are concurrent. Show that the perpendiculars from \(D, E, F\) on \(BC, CA, AB\) respectively are also concurrent.

Construction: Let \(AA', BB', CC'\) be the three concurrent \(\perp s\) from \(A, B, C\) on \(EF, DF, DE\) respectively and \(DD', EE', FF'\) the other
three \perp s from $D$, $E$, $F$ on $BC$, $CA$, $AB$ respectively. Join $AE$, $AF$, $BD$, $BF$, $CD$, $CE$ (Fig. 67).

**Proof:** According to Problem 2.20, since $AA'$, $BB'$ and $CC'$ are concurrent, \( (B'D^2 - B'F^2) + (A'F^2 - A'E^2) + (C'E^2 - C'D^2) = 0 \). Now, in the $\triangle BDF$, \( (B'D^2 - B'F^2) = (BD^2 - BF^2) \). Similarly in $\triangle AEF$, $CDE$, \( (A'F^2 - A'E^2) = (AF^2 - AE^2) \) and \( (C'E^2 - C'D^2) = (CE^2 - CD^2) \). Adding yields \( (BD^2 - BF^2) + (AF^2 - AE^2) + (CE^2 - CD^2) = 0 \). By rearranging, \( (BD^2 - CD^2) + (AF^2 - BF^2) + (CE - AE^2) = 0 \), which is equal to \( (D'B^2 - D'C^2) + (F'A^2 - F'B^2) + (E'C^2 - E'A^2) = 0 \). Therefore, according to the converse of Problem 2.20, the perpendiculars $DD'$, $EE'$, $FF'$ are concurrent.

**Miscellaneous Exercises**

1. If a square is described on a side of a rhombus, show that the area of this square is greater than that of the rhombus.

2. Of all parallelograms on the same base and of the same area, that which is rectangular has the smallest perimeter.

3. $ABCD$ is a square. $E$, $F$, $G$ are the middle points of the sides $AB$, $BC$, $CD$ respectively. Prove that $\triangle AFG = \triangle DEF$. (Both = \( \frac{1}{2} ABCD \).)

4. $APB$, $ADQ$ are two straight lines such that the triangles $PAQ$, $BAD$ are equal. If the parallelogram $ABCD$ be completed, and $BQ$ joined cutting $CD$ in $R$, show that $CR = AP$.

5. $ABC$ is a triangle. $DE$ is drawn parallel to $BC$ meeting $AB$, $AC$ or these produced in $D$, $E$ respectively. If $BE$, $CD$ intersect in $G$, show by means of Problem 2.2 that $\triangle AGD = \triangle AEG$ and that $AG$, produced if necessary, bisects $BC$. 

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**Figure 67**

![Diagram of geometric shapes and lines](image-url)
6. Construct a parallelogram equal and equiangular to a given parallelogram and having one of its sides equal to a given straight line.

7. In the right-angled triangle $ABC$, $ABDE$, $ACFG$ are the squares on the sides $AB$, $AC$ containing the right angle. $DH$, $FK$ are the perpendiculars from $D$ and $F$ on $BC$ produced. Prove that $\triangle DHB$, $CFK$ are together equal to $\triangle ABC$. (Draw $AN \perp BC$.)

8. $ABC$ is a triangle and $D$ is the middle of the base $BC$. If $E$ is any point inside the triangle, then $\triangle ABE - \triangle ACE = 2 \triangle ADE$.

9. $ABCD$, $AECE$ are parallelograms between the same parallels $EAD$, $BCF$. $FG$ is drawn parallel to $AC$, meeting $BA$ on $G$. Prove that the $\triangle$s $ABE$, $ADG$ are equal.

10. $F$, $D$ are two points in the side $AC$ of a triangle $ABC$ such that $FC$ is equal to $AD$. $FG$, $DE$ are drawn parallel to $AB$ meeting $BC$ in $G$, $E$. Show that the $\triangle$s $ADE$, $AGF$ are equal.

11. Convert a given quadrilateral $ABCD$ to another quadrilateral $ABCE$ of equal area so that the angle $BAE$ will be equal to a given angle.

12. $ABC$ is a triangle and $P$ is any point on $AC$. Divide the triangle into two equal parts by a straight line through $P$ parallel to $BC$.

13. If straight lines be drawn from any point $P$ to the vertices of a parallelogram $ABCD$, prove that $\triangle PBD$ is equal to the sum or difference of $\triangle$s $PAB$, $PCB$ according as $P$ is (a) outside; (b) inside the angle $ABC$ or its vertically opposite angle.

14. Construct a parallelogram equal to a given parallelogram and having its sides equal to two given straight lines. Show when the problem is impossible.

15. Find a point $O$ inside a triangle $ABC$ such that the triangles $OAB$, $OBC$, $OCA$ are equal.

16. $ABCD$ is a parallelogram. From $A$ is drawn a straight line $AEF$ cutting $BC$ in $E$ and $DC$ produced in $F$. Prove that $\triangle BEF = \triangle DCE$.

17. $ABCD$ is a quadrilateral and $E$, $F$ are the middle points of $AC$, $BD$ respectively. Prove that if the diagonals meet in $O$, then $(\triangle AOB + \triangle COD) - (\triangle AOD + \triangle BOC) = 4 \triangle EOF$.

18. Prove that the parallelogram formed by drawing straight lines through the vertices of a quadrilateral parallel to its diagonals is double the quadrilateral. Hence prove that two quadrilaterals are equal if their diagonals are equal and contain equal angles.

19. $ABCD$ is a rectangle and $E$, $F$ are any two points on $BC$, $CD$ respectively. Show that $\triangle AEF = ABCD - BE \cdot DF$. (Through $E$, $F$ draw $\parallel$s to $CD$, $BC$ to intersect in $G$. Join $AG$.)

20. $ABCD$ is a parallelogram. If $AC$ is bisected in $O$ and a straight line $MON$ is drawn to meet $AB$, $CD$ in $M$, $N$ respectively, and $OR$ parallel to $AB$ meets $AN$ in $R$, then the $\triangle$s $ARM$, $CRN$ are equal.
21. On the sides $BC$, $AC$ of a right-angled triangle $ABC$ at $A$, squares $BCDE$, $ACFG$ are described. If $L$ is any point taken on $AC$, show that $\triangle DCL = \triangle FCL$ and hence deduce that $\triangle ADC + \triangle AEB = \frac{1}{2} BC^2$.

22. Find the area of an isosceles trapezoid having the base angle equal to $60^\circ$, one of its non-parallel sides 12 inches, and its altitude equal to half the median. (Answer: 216 square inches.)

23. Construct a triangle equal in area to a given triangle $ABC$, so that (a) it will have a given altitude; (b) it will have a base on a part of $BC$ or on $BC$ produced.

24. Show that any line drawn from the mid-point of the median of any trapezoid and terminated by the parallel sides bisects the trapezoid into two equal parts.

25. $ABCD$ is a quadrilateral and $E$, $F$ are the middle points of $AB$, $CD$. Show that if $AF$, $DE$ intersect in $G$ and $CE$, $BF$ intersect in $H$, then the $\triangle EGFH = \triangle AGD + \triangle BHC$.

26. Inscribed a triangle $CDE$ inside a given triangle $ABC$ so that $CD$ will lie on the base $BC$, its vertex $E$ on $AB$, and be equal to half the triangle $ABC$.

27. $P$ is a point inside a parallelogram $ABCD$ such that the area of the quadrilateral $PBCD$ is twice that of the figure $PBAD$. Find the locus of $P$. (Join $AC$, $BD$ cutting in $O$. Trisect $AO$ in $P$, so that $AP = 2 PO$.)

28. If through the vertices of a triangle $ABC$ there be drawn three parallel straight lines $AD$, $BE$, $CF$ to meet the opposite sides or sides produced in $D$, $E$, $F$, then the area of the triangle $DEF$ is double that of $ABC$.

29. $ABC$ is a triangle. $D$, $E$, $F$ are points on $BC$, $CA$, $AB$ respectively such that $BD$ is twice $DC$, $CE$ twice $EA$, and $AF$ twice $FB$. Prove that the triangle $DEF$ is $\frac{1}{4}$ of the triangle $ABC$.

30. The areas of all quadrilateral figures, the sides of which have the same points of bisection, are equal.

31. Show by a figure how to divide a triangle by a line so that its two parts may be made to coincide with a parallelogram on the same base as the triangle and of half its altitude.

32. The diagonals of a trapezoid intersect in the straight line joining the middle points of its parallel sides.

33. In a given triangle inscribe a parallelogram equal to half the triangle so that one side is in the same straight line with one side of the triangle and has one end at a given point in that side.

34. $ABC$ is a right-angled triangle at $B$. On $BC$ describe an equilateral triangle $BCD$ outside the triangle $ABC$ and join $AD$. Show that $\triangle BCD = \triangle ACD - \triangle ABD$. (Bisect $BC$ in $F$. Join $AF$, $DF$.)

35. Two parallelograms $ACBD$, $A'C'B'D'$ have a common angle $C$. Prove that $DD'$ passes through the intersection of $A'B$ and $AB'$.

36. Construct a triangle equal in area to a given triangle so that its vertex will be equidistant from two given intersecting straight lines.
37. Prove that the area of a trapezoid is equal to the rectangle contained by either of the non-parallel sides and the distance between that side and the middle point of the other side.

38. Describe a triangle equal in area to the sum or difference of two given triangles.

39. If $O$ be any point in the plane of a parallelogram $ABCD$ and the parallelograms $OAEB$, $OBFC$, $OCGD$, $ODHA$ be completed, then $EFGH$ is a parallelogram whose area is double that of $ABCD$.

40. Through any point in the base of a triangle two straight lines are drawn in given directions terminated by the sides of the triangle. Prove that the part of the triangle cut off by them will be a maximum when the straight line joining their extremities is parallel to the base.

41. Through the middle point of the side $AB$ of the triangle $ABC$, a line is drawn cutting $CA$, $CB$ in $D$, $E$ respectively. A parallel line through $C$ meets $AB$ or $AB$ produced in $F$. Prove that the triangles $ADF$, $BEF$ are equal.

42. $D$, $E$, $F$ are the middle points of the sides $BC$, $CA$, $AB$ of a triangle $ABC$. $FG$ is drawn parallel to $BE$ meeting $DE$ produced in $G$. Show that the sides of the triangle $CFG$ are equal to the medians of the triangle $ABC$. Hence show also that the area of the triangle $CFG$ which has its sides respectively equal to the medians of the triangle $ABC$ is $\frac{1}{3}$ that of $ABC$.

43. $D$, $E$, $F$ are the middle points of the sides $BC$, $CA$, $AB$ of a triangle. Any line through $A$ meets $DE$, $DF$ produced if necessary in $G$, $H$ respectively. Show that $CG$ is parallel to $BH$.

44. Two equal and equiangular parallelograms are placed so as to have a common angle $BAC$. $P$, $Q$ are the intersections of their diagonals. If $PQ$ produced cut $AB$, $AC$ in $M$, $N$ respectively, show that $PM = QN$. (Let $ABDC$, $AGEF$ be the ,; let $G$ lie between $A$, $B$ and $C$ between $A$, $F$ Join $DE$ and produce it to meet $AB$, $AC$ produced in $R$, $S$. Join $GC$, $GD$, $CE$.)

45. $D$, $E$, $F$ are the middle points of the sides $BC$, $CA$, $AB$ of the triangle $ABC$. Through $A$, $B$, $C$ are drawn three parallel straight lines meeting $B'C'$, $C'A'$, $A'B'$ respectively in $a$, $b$, $c$. Prove that $bc$ passes through $A$, $ca$ through $B$, and $ab$ through $C$ and that the triangle $abc$ is half the triangle $ABC$. (Join $Ba$, $ac$.)

46. On the smaller base $DC$ produced of a trapezoid $ABCD$ find a point $P$ so that if $PA$ is joined, it will divide the trapezoid into two equal parts. (Convert $ABCD$ into an equal triangle $ADG$ where $G$ is on $DC$ produced using Problem 2.4. Bisect $DG$ in $Q$ and draw $QM \parallel BG$ and cut $BC$ in $M$. Hence $AM$ produced meets $DG$ in $P$.)

47. Squares are described on the sides of a quadrilateral and the adjacent corners of the squares joined so as to form four triangles. Prove that two of these triangles are together equal to the other two.

48. Construct a square that will be $\frac{1}{3}$ of a given square.
49. If $ABCD$ is a straight line, prove that $AC \cdot BD = AB \cdot CD + BC \cdot AD$.

50. If two equal straight lines intersect each other anywhere at right angles, the quadrilateral formed by joining their extremities is equal to half the square on either line.

51. If $R$ is the middle point of the straight line $PQ$. $PL$, $QM$, $RN$ are drawn perpendicular to another straight line meeting it in $L$, $M$, $N$ respectively. Show that the figure $PLMQ = LM \cdot RN$.

52. If in an isosceles triangle a perpendicular be let fall from one of the equal angles on the opposite side, the square on this perpendicular is greater or less than the square on the line intercepted between the other equal angle and the perpendicular, by twice the rectangle contained by the segments of that side, according as the vertical angle of the triangle is acute or obtuse.

53. $BA$ is divided in $C$ so that the square on $AC$ is equal to the rectangle $AB$, $BC$ and produced to $D$ so that $AD$ is twice $AC$. Show that the square on $BD$ is five times the square on $AB$.

54. An equilateral triangle is described, one of whose vertices is at the angle $B$ of another equilateral triangle $ABC$, and whose opposite side $PQ$ passes through $C$. Prove that $BP^2 = AB^2 + PC \cdot QC$.

55. On a given straight line describe a rectangle which will be equal to the difference of the squares on two given straight lines.

56. Any rectangle is half the rectangle contained by the diagonals of the squares on its adjacent sides.

57. $A$ and $B$ are fixed points, $CD$ a fixed straight line of indefinite length. Find a point $P$ in $CD$ such that $(PA^2 + PB^2)$ is a minimum.

58. The squares on the diagonals of any quadrilateral are double the squares on the lines joining the middle points of the opposite sides. (See Problem 2.31.)

59. Show that the sum of the squares on the distances of any point from the angular points of a parallelogram is greater than the sum of the squares on two adjacent sides by four times the square on the line joining the point to the point of intersection of the diagonals.

60. $ABC$, $DEF$ are two triangles having the two sides $AB$, $AC$ equal to the two sides $DE$, $DF$, each to each, and having the angles $BAC$, $EDF$ supplementary. Show that $BC^2 + EF^2 = 2 (AB^2 + AC^2)$.

61. Two squares $ABCD$, $A'B'C'D'$ are placed with their sides parallel, $AB$ parallel to $A'B'$, and so on. Prove that $AA'^2 + CC'^2 = BB'^2 + DD'^2$.

62. On the sides of a quadrilateral, squares are described outward, forming an eight-sided figure by joining the adjacent corners of consecutive squares. Prove that the sum of the squares on the eight sides with twice the squares on the diagonals of the quadrilateral is equal to five times the sum of the squares on its sides.
63. The diagonal $AC$ of a square $ABCD$ is produced to $E$ so that $CE$ is equal to $BC$. Show that $BE^2 = AC \cdot AE$.

64. $ABCD$ is a parallelogram whose diagonals intersect in $O$. $OL$, $OM$ are drawn perpendicular to $AB$, $AD$ meeting them in $L$, $M$ respectively. Show that $AB \cdot AL + AD \cdot AM = 2 AO$.

65. $AB$, $AC$ are the equal sides of an isosceles triangle $ABC$. $BD$ is drawn perpendicular to $AB$ meeting $AC$ produced in $D$ and the bisector of the angle $A$ meets $BD$ in $E$. Prove that the square on $AB$ is equal to the difference of the rectangles $DA$, $AC$ and $DB$, $BE$.

66. From the vertices of a triangle $ABC$, three perpendiculars $AD$, $BE$, $CF$ are drawn on any straight line outside $ABC$. Show that the perpendiculars from $D$, $E$, $F$ on $BC$, $CA$, $AB$ are concurrent. (Join $AE$, $AF$, $BD$, $BF$, $CD$, $CE$ and apply Problem 2.20, making use of Problem 2.32.)

67. $ABCD$ is a trapezoid having sides $AB$, $CD$ parallel and the sides $AD$, $BC$ perpendicular to each other. $E$, $F$, $G$, $H$ are the middle points of $AB$, $CD$, $AD$, $BC$ respectively. Prove that the difference of the squares on $CD$, $AB$ is equal to four times the rectangle $EF$, $GH$.

68. When the perimeter of a parallelogram is given, its area is a maximum when it is a square. (If the two sides are given, the area is a maximum when the parallelogram is a rectangle. Hence the question resolves itself to: Of all rectangles with the same perimeter, the square has the greatest area.)

69. If $P$ be the orthocenter of the triangle $ABC$, show that (a) $AP^2 + BC^2 = BP^2 + AC^2 = CP^2 + AB^2$; (b) $AP \cdot BC + BP \cdot AC + CP \cdot AB = 4 \triangle ABC$.

70. Prove that three times the sum of the squares on the sides of a triangle are equal to four times the sum of the squares on its medians.

71. $G$ is the centroid of the triangle $ABC$ and $P$ any point inside or outside the triangle. Show that (a) $AB^2 + BC^2 + AC^2 = 3 (AG^2 + BG^2 + CG^2)$; (b) $PA^2 + PB^2 + PC^2 = AG^2 + BG^2 + CG^2 + 3 PG^2$.

72. Divide the straight line $AB$ into two parts at $C$ so that the square on $AC$ may be equal to twice the square on $BC$ and prove that the squares on $AB$, $BC$ are together equal to twice the rectangle $AB$, $AC$.

73. $ABC$ is any triangle. Find a point $P$ on the base $BC$ so that the difference of the squares on $AB$, $AC$ will be equal to $BC \cdot BP$. (Bisect $BC$ in $D$ and draw $AE \perp BC$. Take $BP$ on $BC = 2 DE$.)

74. $ABCD$ is a quadrilateral in which $AC = CD$, $AD = BC$, and $\angle ACB$ is the supplement of $\angle ADC$. Show that $AB^2 = BC^2 + CD^2 + DA^2$.

75. $D$ is the foot of the perpendicular from $A$ on the side $BC$ of the triangle $ABC$; $E$ is the middle point of $AC$. Prove that the square on $BE$ is equal to the sum or difference of the square on half $AC$ and the rectangle $BC$, $BD$, according as the angle $B$ is acute or obtuse.
76. Find the locus of a point such that the sum of the squares on its distances from (a) two given points; (b) three given points, may be constant.

77. Of all parallelograms inscribed in a given rectangle, that whose vertices bisect the sides of the rectangle has the least sum of squares of sides.

78. \(AB, CD\) are two straight lines, and \(F, G\) the middle points of \(AC, BD\) respectively. \(H, K\) are the middle points of \(AD, BC\) respectively. Prove that \(AB^2 + CD^2 = 2(FG^2 + HK^2)\). (Join \(FH, HG, GK, KF\).)

79. Prove that the locus of a point such that the sum of the squares on its distances from the vertices of a quadrilateral is constant is a circle the center of which coincides with the intersection of the lines joining the middle points of opposite sides of the quadrilateral.

80. On \(AB\) describe the square \(ABCD\), bisect \(AD\) in \(E\), and join \(EB\). From \(EB\) cut off \(EF\) equal to \(EA\) and from \(BA\) cut off \(BG\) equal to \(BF\). Show that the rectangle \(AB, AG\) is equal to the square on \(BG\).

81. Divide a straight line into two parts so that the square on the whole line with the square on one part shall be equal to three times the square on the other part.

82. Construct a right-angled triangle with given hypotenuse such that the difference of the squares on the sides containing the right angle may be equal to the square on the perpendicular from the right angle on the hypotenuse.

83. In Fig. 59(i), \(GD\) is joined cutting \(AB, HL\) in \(g, d\) respectively. Show that \(Gg = Dd\). If \(CK, DB\) produced meet in \(N\), prove also that \(CK^2 + CN^2 = 5AB^2\).

84. \(ABC\) is a right-angled triangle, \(BAC\) being the right angle. Any straight line \(AO\) drawn from \(A\) meets \(BC\) in \(O\). From \(B\) and \(C\), \(BM, CN\) are drawn perpendicular to \(AO\) or \(AO\) produced. Prove that the squares on \(AM, AN\) are equal to the squares on \(OM, ON\) with twice the rectangle \(BO, OC\).

85. Produce \(AB\) to \(C\) so that the rectangle \(AB, AC\) may be equal to the square on \(BC\).

86. \(AB\), one of the sides of an equilateral triangle \(ABC\), is produced to \(D\) so that \(BD\) is equal to twice \(AB\). Prove that the square on \(CD\) is equal to seven times the square on \(AB\).

87. Two right-angled triangles \(ACB, ADB\) have a common hypotenuse \(AB, AC, BD\) meet in \(E, AD, BC\) in \(F\). Prove that the rectangles \(AE, EC\) and \(BE, ED\) are equal, also the rectangles \(AF, FD\) and \(BF, FC\).

88. \(ABC\) is a triangle having the angle \(A\) equal to half a right angle. \(M, N\) are the feet of the perpendiculars from \(B, C\) on the opposite sides. Prove that \(BC^2 = 2MN^2\).

89. Given the sum of two lines and also the sum of the squares described on them. Obtain by a geometric construction the lines themselves.
90. If a straight line be bisected and produced to any point, the square on the whole line is equal to the square on the produced part with four times the rectangle contained by half the line bisected and the line made up of the half and the produced part.

91. Of all parallelograms of equal perimeter, the sum of the squares on the diagonals is least in those whose sides are equal.

92. $ABCD$ is a square and $E$ is any point in $DC$. On $AE$, $BE$ outside the triangle $AEB$ are described squares $APQE$, $BRSE$. Prove that the square on $QS$ is less than five times the square on $AB$ by four times the rectangle $DE$, $DC$.

93. If in a quadrilateral the sum of the squares on one pair of opposite sides is equal to the sum of the squares on the other pair, the lines joining the middle points of opposite sides are equal.

94. On the same base $AB$ and on opposite sides of it are described a right-angled triangle $AQB$, $Q$ being the right angle, and an equilateral triangle $APB$. Show that the square on $PQ$ exceeds the square on $AB$ by twice the rectangle contained by the perpendiculars from $P$, $Q$ on $AB$. (Draw $PM$, $QN$ $\perp$ $AB$, then $AM = MB = MQ$. Draw $QR \parallel AB$ meeting $PM$ produced in $R$.)

95. Show how to inscribe in a given right-angled isosceles triangle a rectangle equal to a given rectilinear figure. When is it impossible to do this?
CHAPTER 3

CIRCLES AND TANGENCY

Theorems and Corollaries

DIAMETERS, CHORDS, AND ARCS IN CIRCLES

3.56. (i) The diameter of a circle which bisects a chord is perpendicular to the chord. (ii) The diameter which is perpendicular to a chord bisects it. (iii) The perpendicular bisector of any chord contains the center.

Corollary. A circle is symmetric with regard to any diameter.

3.57. One circle and only one, can be drawn through three given points not in the same straight line.

Corollary 1. Circles which have three common points coincide.

Corollary 2. If $O$ is a point within a circle from which three equal straight lines $OA$, $OB$, $OC$ can be drawn to the circumference, then $O$ is the center of the circle.

3.58. In the same circle or in equal circles: (i) If two arcs subtend equal angles at the center, they are equal. (ii) Conversely, if two arcs are equal, they subtend equal angles at the center.

3.59. In the same circle or in equal circles: (i) If two arcs are equal, then the chords of the arcs are equal. (ii) Conversely, if two chords are equal, then the minor arcs which they cut off are equal and so are the major arcs. (iii) Equal arcs, or equal chords, determine equal sectors and equal segments of the circle.

3.60. In the same circle or in equal circles: (i) If two chords are equal, they are equidistant from the center. (ii) Conversely, if two chords are equidistant from the center they are equal.

3.61. Two chords of a circle, which do not both pass through the center, cannot bisect each other. Either chord may be bisected by the other, but they cannot both be bisected at their point of intersection.

Corollary. The diagonals of any parallelogram inscribed in a circle intersect in the center of the circle.

3.62. If two circles intersect, they cannot have the same center.

3.63. One circle cannot cut another in more than two points.

3.64. The diameter is the greatest chord of a circle and of all others the chord which is nearer to the center is greater than one more remote, and the greater is nearer the center than the less.
ANGLES SUBTENDED BY ARCS IN CIRCLES

3.65. The angle which an arc of a circle subtends at the center is double that which it subtends at any point on the remaining part of the circumference.

3.66. (i) Angles in the same segment of a circle are equal. (ii) The angle in a segment which is greater than a semicircle is less than a right angle. (iii) The angle in a semicircle is a right angle. (iv) The angle in a segment which is less than a semicircle is greater than a right angle.

3.67. In the same circle or in equal circles, equal arcs subtend equal angles at the circumference and equal angles at the center.

TANGENTS AND TOUCHING CIRCLES

3.68. One tangent, and only one, can be drawn to a circle at any point on the circumference, and this tangent is perpendicular to the radius through the point of contact.

COROLLARY 1. The straight line joining the center of a circle to the point of contact of a tangent is perpendicular to the tangent.

COROLLARY 2. The perpendicular to a tangent to a circle at the point of contact passes through the center.

3.69. If two tangents are drawn to a circle from an external point: (i) The lengths of the tangents from the external point to the points of contact are equal. (ii) They subtend equal angles at the center of the circle. (iii) They make equal angles with the straight line joining the given point to the center.

3.70. If A is any point inside or outside a circle with center O, and OA or produced cuts the circumference in B, then AB is the shortest distance from the point A to the circumference of the circle.

3.71. Two circles cannot touch at more than one point, and if they touch, the point of contact lies in the straight line joining the centers, or in that line produced.

3.72. If through an extremity of a chord in a circle a straight line is drawn: (i) If the straight line touches the circle, the angles which it makes with the chord are equal to the angles in the alternate segments. (ii) Conversely, if either of the angles which the straight line makes with the chord is equal to the angle in the alternate segment, the straight line touches the circle.

CYCLIC POLYGONS

3.73. If a quadrilateral is such that one of its sides subtends equal angles at the extremities of the opposite side, the quadrilateral is cyclic.

COROLLARY 1. The locus of points at which a finite straight line AB subtends a given angle consists of the arcs of two segments of circles on AB as base, each containing the given angle.
Corollary 2. The locus of points at which a given straight line $AB$ subtends a right angle is the circle on $AB$ as diameter.

Corollary 3. The straight line joining the middle point of the hypotenuse of a right-angled triangle to the vertex is equal to half the hypotenuse.

3.74. The opposite angles of a quadrilateral inscribed in a circle are supplementary.

Corollary. If a side of a quadrilateral inscribed in a circle is produced, the exterior angle is equal to the interior opposite angle.

3.75. If two opposite angles of a quadrilateral are supplementary, the quadrilateral is cyclic.

Corollary. If one side of a quadrilateral is produced and the exterior angle so formed is equal to the interior opposite angle, then the quadrilateral is cyclic.

Nine-point Circle

3.76. The circle through the middle points of the sides of a triangle passes through (i) the feet of the perpendiculars from the vertices of the triangle on the opposite sides; (ii) the middle points of the lines joining the orthocenter to the vertices. This is called the nine-point circle.

Corollary. The radius of the nine-point circle of a triangle is equal to half of the circumcircle.

3.77. (i) The center of the nine-point circle of any triangle is the middle point of the line joining the circumcenter and orthocenter of the triangle. (ii) The centroid is a point of trisection of this line.

Squares and Rectangles Related to Circles

3.78. (i) If two chords of a circle intersect at a point, either inside or outside the circle, the rectangle contained by the segments of the one is equal to that contained by the segments of the other. (ii) When this point is outside the circle, each of these rectangles is equal to the square on the tangent from the point of intersection of the chords to the circle.

3.79. If two straight lines intersect, or both being produced intersect, so that the rectangle contained by the segments of the one is equal to that contained by the segments of the other, the extremities of the lines are cyclic. This is the converse of Th. 3.78(i).

3.80. If a chord $AB$ of a circle is produced to any point $P$ and from this point a straight line $PQ$ is drawn to meet the circle, such that the square on $PQ$ is equal to the rectangle $AP \cdot BP$, then the straight line $PQ$ touches the circle at $Q$.

3.81. In a right-angled triangle $ABC$ in which $A$ is the right angle and $BC$ is the hypotenuse, if $AP$ is drawn perpendicular to the hypotenuse $BC$ then: (i) $AB^2 = BP \cdot BC$; (ii) $AC^2 = CP \cdot BC$; (iii) $AP^2 = PB \cdot PC$. 
Corollary. If a line $BC$ is divided internally in $P$, then $BC^2 = BP \cdot BC + CP \cdot BC$. If it is divided externally in $P$ such that $BP$ is greater than $PC$, then $BC^2 = BP \cdot BC - CP \cdot BC$.

Solved Problems

3.1. $O$ is the center of a circle whose circumference passes through the vertices of an equilateral triangle $ABC$. If $P$ is any point taken on the circumference and $AD, BE, CF$ are three perpendiculars from $A, B, C$ on the tangent from $P$ to the circle, show that the sum of these perpendiculars is equal to twice the altitude of the $\triangle ABC$.

Construction: Draw $AM \perp BC$; then produce it to meet the circle in $N$. Draw the perpendiculars $MK, NL$ on the tangent from $P$ and join $OP$ (Fig. 68).

![Figure 68](image)

Proof: Since $AM$ is an altitude in the equilateral $\triangle ABC$, it bisects the base $BC$ and hence passes through the center of the $\circ$. Also, $OP \perp$ tangent $DPEF$ (Th. 3.68, Cor.1). Since $\triangle ABC$ is equilateral, $\therefore OM = MN = \frac{1}{2} OP$. $\therefore M$ is the mid-point of $ON$. $\therefore BCFE$ is a trapezoid, $\therefore BE + CF = 2 MK$ (Th. 1.29). Similarly, in the trapezoid $OPLN$, $OP + NL = 2 MK$. Hence $BE + CF = OP + NL$. Now, in a similar way, the trapezoid $ADLN$ gives $AD + NL = 2 OP$. Adding $OP$ to each side, $\therefore AD + NL + OP = 3 OP$. $\therefore AD + BE + CF = 3 OP = 2 AM$.

3.2. Through one of the points of intersection of two given circles, draw a straight line terminated by the two circles which will be equal to a given straight line.

Construction: Let $A$ be one of the points of intersection of $\circ$s
ABC, ABD; O, O' their centers. On OO' as diameter describe semi-circle OEO'. With center O' and distance equal to half the given straight line, describe an arc of ⊙ cutting the semi-circle in E. Join O'E and through A draw CAD || O'E. Join OE and produce it to meet CD in F. Draw O'G ⊥ CD (Fig. 69).

**Proof:** ∠OEO' is right [Th. 3.66 (iii)] and CD is || O'E, ∴ ∠OFA is right. ∴ OF is || O'G. Hence EFGO' is a rectangle. ∴ FG = EO'. But AC is double of AF and AD double of AG. ∴ CD is double of FG, i.e., of EO'. ∴ CD = given straight line.

**Corollary.** Because CD is twice O'E, CD will be a maximum when O'E is a maximum, i.e., when O'E coincides with OO'. Therefore, the greatest straight line which can be drawn through A is parallel to the line joining the centers of the circles and is double that line. We shall leave for the student the question which is the shortest given straight line that can be used subject to the given conditions.

3.3. Construct a triangle given a vertex and (i) the circumscribed circle and center of its incircle; (ii) the circumscribed circle and the orthocenter.

**Construction:** (i) Consider the vertex A on the given circumference. Join AP, P being the given center of the incircle of Δ. Produce AP to meet the circumference of the given circumscribed ⊙O in Q. From Q draw the two chords QB, QC equal to QP; then ABC is the required Δ (Fig. 70).

**Proof:** ∵ QB = QP (construct), ∴ ∠QBP = ∠QPB. ∴ ∠QBC + ∠CBP = ∠PBA + ∠PAB. But, since PA bisects ∠A, ∴ ∠PAB = ∠PAC = ∠QBC (Th. 3.67). ∴ ∠CBP = ∠PBA; i.e., PB bisects ∠B, and similarly PC bisects ∠C. Hence P is the incenter of ΔABC and therefore ABC is the required Δ.

**Construction:** (ii) Circumcircle O and orthocenter D are given. From the given vertex A on the given circumference, join AD and
produce it to meet the circumference in \( G \). Bisect \( DG \) in \( F \) and draw \( BFC \perp ADG \). Then \( ABC \) is the required \( \triangle \).

**Proof:** \( \triangle s BDF, BGF \) are congruent. \( \therefore \angle DBF = \angle GBF \). \( \therefore \angle GBF = \angle GAC \). \( \therefore \angle DBF = \angle GAC \) or \( \angle FAE \). \( \therefore \triangle ABFE \) is cyclic (Th. 3.73). \( \therefore \angle BFA = \angle BEA = \) right angle. \( \therefore BE \perp AC \). \( \therefore AF \perp BC \). \( \therefore D \) is the orthocenter of \( \triangle ABC \).

3.4. \( I \) is the incenter of a triangle \( ABC \) and the circumcircle of triangle \( BIC \) cuts \( AB, AC \) in \( D, E \) respectively. Show that (i) \( AB = AE \) and \( AC = AD \). (ii) \( I \) is the orthocenter of the triangle formed by joining the circumcenters of \( \triangle s BIC, CIA, AIB \). (iii) The circumcircle of \( \triangle ABC \) passes through the circumcenters of \( \triangle s BIC, CIA, AIB \).

**Construction:** Let \( P, Q, R \) be the circumcenters of \( \triangle s BIC, CIA, AIB \) respectively. Join \( IP, IQ, IR \) and produce them to meet the \( \bigcirc s \) on \( BIC, CIA, AIB \) in \( F, G, H \). Join \( FG, GH, HF, ID, IE, PB, PC \) (Fig. 71).

**Proof:** (i) \( \therefore BI, AI \) bisects \( \angle s B, A \) respectively, \( \therefore \angle DBI = \angle IBC = \angle IEC \). Therefore, \( \triangle s AIB, AIE \) are congruent. \( \therefore AB = AE \). Similarly, \( AD = AC \).

(ii) Since \( P \) is the center and \( IF \) is the diameter of \( \bigcirc BIC \), \( \therefore \angle ICF = \) right angle [Th. 3.66 (iii)]. Also, in \( \bigcirc CIA, \angle ICG = \) right angle. Hence \( \angle ICF + \angle ICG = 2 \) right angles. \( \therefore FCG \) is a straight line (Th. 1.2). Similarly, \( GAH, HBF \) are straight lines. \( \therefore P, Q \) are the mid-points of \( IF, IG \) (since they are centers), \( \therefore PQ \parallel FG \). \( \therefore IC \perp FCG \), \( \therefore PQ \perp IC \). Similarly, \( QR \perp IA \) and \( RP \perp IB \). Hence \( I \) is the orthocenter of \( \triangle PQR \), and \( AIPF, BIQG, CIRH \) are straight lines.

(iii) \( \therefore PB = PI = PC \) and \( PR, PQ \perp s IB, IC \). \( \therefore RP, QP \) bisect \( \angle s IPB, IPC \). \( \therefore \angle QPC = \frac{1}{2} \angle CPI \); i.e., \( \angle CPA = \frac{1}{2} \angle CBA \).
Problems and Solutions in Euclidean Geometry

Figure 71

= \angle QBC. \therefore \text{fig. } PBQC \text{ is cyclic (Th. 3.73). Similarly, fig. } BCQR \text{ is cyclic. Therefore, } BPCQAR \text{ is cyclic or the circumcircle of } \triangle ABC \text{ passes through } P, Q, R.

Note: The points \( F, G, H \) are the centers of the external \( \odot s \) of \( \triangle ABC \). Also the circumcircle of \( \triangle ABC \), which passes through \( P, Q, R \), is the nine-point \( \odot \) of \( \triangle FGH \) (Th. 3.76).

3.5. \( A, B \) are the two points of intersection of three circles. From \( A \) a straight line is drawn to meet the circles in \( D, E, F \). If the tangents from \( D, E \) meet in \( P \), those from \( E, F \) meet in \( Q \) and those from \( F, D \) meet in \( R \), show that \( PBQR \) is cyclic.

Construction: Join \( BD, BA, BE, BF, BR \) (Fig. 72).

Proof: \( \because PE, PD \) are tangents to the two \( \odot s \), \( \therefore \angle PED = \angle EBA. \angle PDE = \angle DBA \) [Th. 3.72 (i)]. Hence, \( \angle PED + \angle PDE = EBD. \) Since, in the \( \triangle PDE, \angle PED + \angle PDE + \angle DPE = 2 \) right angles, \( \therefore \angle EBD + \angle DPE = 2 \) right angles. \( \therefore DBEP \) is cyclic. Similarly, \( DBFR \) is cyclic. Therefore, \( \angle BDE = \angle BPE \) and \( \angle BDF \) or \( \angle BDE \).
3.6. Describe a circle of given radius to touch a given straight line so that the tangents drawn to it from two given points on the straight line will be parallel. How many solutions are there to this problem and when is it impossible to describe this circle?

CONSTRUCTION: Let PQ be the given straight line and B, C the two given points on it. On BC as diameter describe a circle. At distance from BC = given radius, draw a circle cutting the circle in O, O'. Draw OA ⊥ BC; then OA = given radius. Then, O with O as center and OA as radius will be the required circle (Fig. 73).

Proof: Draw BE, CD tangents to circle. ∴ BA, BE are tangents to circle, ∴ OB bisects ∠ABE. Similarly, OC bisects ∠ACD. Since
$\angle BOC = \text{right angle}$ [Th. 3.66(iii)], $\therefore$ in $\triangle BOC$, $\angle OBC + \angle OCB = \text{right angle}$. Hence $\angle ABE + \angle ACD = 2 \text{ right angles}$. $\therefore BE \parallel CD$. Therefore $\odot$ with $O$ as center and $OA$ the given radius is the required $\odot$.

(i) In this case there are four solutions, since the $\parallel$ line to $BC$ at the required radius will cut $\odot$ in two points $O, O'$ on each side of the line $PQ$.

(ii) In the case when $OO'$ touches the $\odot$ at one point, say $O$, its distance from $BC = \text{half of } BC$; i.e., the $\odot$ with $O$ as center will be equal to $\odot$ on $BC$ as diameter. The tangents $BE, CD$ will be $\perp BC$ and there will be two $\odot$s, one at each side of $PQ$.

(iii) The solution is impossible, however, if the given radius of the required $\odot$ is greater than half $BC$. In this case, the $\parallel$ line to $BC$ will never cut $\odot$ on $BC$.

3.7. If circles are inscribed inside the six triangles into which a triangle $ABC$ is divided by its altitudes $AD, BE, CF$, then the sum of the diameters of these six circles together with the perimeter of the triangle $ABC$ equals twice the sum of the altitudes of triangle $ABC$.

**Construction:** Let $O_1, O_2$ be the inscribed $\odot$s of $\triangle ABD, ACD$, touching $BC, AD$ in $G, P, H, Q$. Join $O_1G, O_1H, O_2P, O_2Q$ (Fig. 74).

**Figure 74**

Proof: $\because O_1G, O_1H \perp BD, AD$, $\therefore O_1GDH \text{ is a square. } \therefore O_1G + O_1H = GD + DH = \text{diameter of } \odot O_1$. $\therefore AB = BG + AH$, $\therefore \text{diameter of } \odot O_1 + AB = BD + AD$. Similarly, in $\triangle ADC$, $\text{diameter of } \odot O_2 + AC = CD + AD$. Adding yields diameter of $\odot O_1 + \text{diameter of } \odot O_2 = AB + AC = BC + 2AD$ (1). Similarly, diameter of $\odot O_3 + \text{diameter of } \odot O_4 = AB + 2CF$ (2), diameter of $\odot O_5 + \text{diameter of } \odot O_6 = AC + 2BE$ (3). Adding (1) to (3) yields diameters of $\odot$s $O_1, O_2, O_3, \ldots, O_6 + (AB + BC + AC) = 2(AD + BE + CF)$. 
3.8. ABC is a triangle and AD, BE, CF its altitudes. If P, Q, R are the middle points of DE, EF, FD respectively, show that the perpendiculars from P, Q, R on the opposite sides of the triangle ABC, AB, BC, CA respectively, are concurrent.

**Construction:** Let PX, QY, RZ be the three \( \perp \)s on AB, BC, CA respectively. Join PQ, QR, RP. Let O be the orthocenter of \( \triangle ABC \) (Fig. 75).

![Figure 75](image)

**Proof:** O is the point of concurrence of AD, BE, CF, and DEF is the pedal \( \triangle \). \( \therefore \triangle BDO = \triangle BFO = \text{right angle} \), \( \therefore \triangle BDOF \) is cyclic (Th. 3.75). \( \therefore \triangle FDO = \triangle FBO \). Similarly, \( \triangle CDOE \) is cyclic. \( \therefore \triangle EDO = \triangle ECO \). But since \( \triangle FBCE \) is cyclic also, \( \triangle FDO = \triangle EDO \); i.e., AD bisects \( \angle DFE \) of the pedal \( \triangle DEF \). Similarly, BE, CF bisect \( \angle DEF \). Now, since PQ || DF and QY || AD (both \( \perp \)s BC), \( \therefore \angle PQY = \angle ADF \). Since also QR || DE and QY || AD, \( \therefore \angle RQY = \angle ADE \). \( \therefore \angle ADF = \angle ADE \), \( \therefore \angle PQY = \angle RQY \). Hence QY bisects \( \angle PQR \). Similarly, PX, RZ bisect \( \angle QPR, PRQ \). Therefore, PX, QY, RZ are the angle bisectors of \( \triangle PQR \) and consequently they are concurrent (Th. 1.32).

3.9. AB is a fixed diameter in a circle with center O. C, D are two given points on its circumference and on one side of AB. Find a point P on the circumference so that (i) PC, PD will cut equal distances of AB from O whether P is on the same or opposite side of AB as C, D; (ii) PC, PD will cut a given length m of AB.

**Analysis:** (i) Suppose P is the required point. PC, PD or produced will cut AB in Q, R, where OQ = OR. Join CO and produce it to meet the circumference in E, then join ER, ED (Fig. 76). Now, from the congruence of \( \triangle QOC, ORE, \angle QCO = \angle OER \). \( \therefore \angle QCE = \angle PDE \) (Th. 3.66), \( \therefore \) in (i) \( \angle OER = \angle PDE \). \( \therefore \)
\[ \angle OED = \angle DRE \] and in (ii) \[ \angle OER = \angle PDE. \] \[ \therefore \ OE \] touches \( \odot \) circumscribed on \( \triangle RDE \) with center \( O_1 \) [Th. 3.72(ii)]. Hence \( O_1 E \perp OE \) and \( O_1 E = O_1 D \). Therefore,

**Synthesis:** Join \( CO \) and produce it to \( E \). Draw \( EO_1 \perp EOC \) and make an \( \angle EDO_1 = DEO_1 \). Then \( O_1 \) is the center of \( \odot \) touching \( OE \) in \( E \). Draw \( \odot \) with center \( O_1 \) and radius \( O_1 E \) to cut \( AB \) or produced in \( R \). Join \( DR \) and produce it to meet \( \odot \) in the required point \( P \). Join \( CP \) to cut \( AB \) in \( Q \). Now, since \( OE \) touches \( \odot O_1 \), \[ \therefore \) in (ia) \[ \angle OED = \angle DRE. \] \[ \therefore \) \[ \angle OER = \angle PDE = \angle QCO. \] \[ \therefore \ ER \parallel PQ \] and in (ib) \[ \angle OER = \angle EDR. \] \[ \therefore \angle EDP = \angle ECP, \] \[ \therefore \angle OER = \angle ECP. \] \[ \therefore \ ER \parallel PQ. \] Since \( COE \) is a diameter in \( \odot O \), \[ \therefore \ OQ = OR. \]

**Analysis:** (ii) Let \( P \) be the required point. Join \( PC, PD \) cutting \( AB \) in \( Q, R \), where \( QR \) is the given length \( m \). Draw \( CS \) equal and \[ || QR \] (Fig. 77). \[ \therefore \) \( CQRS \) is a \( \square \). \[ \therefore \) \( SR \parallel CQP. \] \[ \therefore \) \( \angle DRS = \angle DPC, \]
since $D, S$ are two fixed points (because $C$ and length $CS$ are given).

**Synthesis**: From $C$ draw $CS \parallel AB$ and equal to the given length $m$ on $AB$. On $SD$, draw a $\bigcirc$ subtending $\angle SRD = \angle DPC$ to cut $AB$ in $R$. Join $DR$ and produce it to meet $\bigcirc O$ in $P$. Therefore, $SR \parallel CP$ and $CQRS$ is a $\square$. \therefore $QR = CS = \text{given length } m$.

**3.10. If the vertex angle $A$ of a triangle $ABC$ is $60^\circ$, prove that if $O$ is the circumcenter, $D$ is the orthocenter, $E$ and $F$ are the centers of the inscribed and escribed circles touching the base $BC$ of the triangle, then $OEDF$ is cyclic. Show also that $O, D$ are equidistant from $E, F$.

**Construction**: Let $AEF$ cut $\bigcirc O$ in $P$. Join $OP$ cutting $BC$ in $Q$. Join also $AO, OB, BE, BP, PD, BF$ (Fig. 78).

![Figure 78](image)

**Proof**: $\angle BOP = 2 \angle BAP$ (Th. 3.65). \therefore $AEFP$ bisects $\angle A$, which $= 60^\circ$, \therefore $\angle BOP = 60^\circ$. Since $OB = OP$, \therefore $\triangle OBP$ is equilateral.

Hence $PB = PO$. \therefore $\angle PEB = \angle EAB + \angle EBA = \angle EAC + \angle EBC = \angle PBC + \angle EBC = \angle EB$, \therefore $PB = PE = PO$. \therefore $BE, BF$ are the internal and external bisectors of $\angle B$, they are $\perp$ to one another. Hence $EBF$ is a right-angled triangle at $B$ and $PB = PE$. \therefore $\angle PBC = \angle PAC = 30^\circ$. Since $\angle PBO = 60^\circ$, \therefore $\angle PBC = \frac{1}{3} \angle PBO$. Therefore, $PQ = QO$. But $AD = 2 QO$ (Problem 1.32). \therefore $AD = \parallel OP$. \therefore $AO = OP$, \therefore fig. $AOPD$ is a rhombus \therefore $PO = PD$.

Since $PO = PE = PB = PF$, \therefore $P$ is the center of $\bigcirc$ passing
through $OEDF$, and also through $B, C$, with $EF$ as diameter. Again, since $AOPD$ is a rhombus, $\therefore$ diagonal $PA$ bisects $\angle OPD$. $\therefore PO = PD$, $\therefore \triangle OPE, DPE$ are congruent. $\therefore EO = ED$. Similarly, $FO = FD$.

3.11. $ABCD$ is a cyclic quadrilateral inscribed in a circle. $E, F, G, H$ are the middle points of the arcs $AB, BC, CD, DA$. $P, Q, R, S$ are the centers of the circles inscribed in the triangles $ABC, BCD, DCA, DBA$ respectively. Show that the figure $PQRS$ is a rectangle whose sides are parallel to $EG, FH$.

**Construction:** Join $AS, BP, AP, PF, BS, SH, CP, PE, DS, SE$ (Fig. 79).

![Figure 79](image)

**Proof:** Let $EG, FH$ meet in $T$. $\angle FTG$ is measured by half the sum of the arcs $FG, EH = \frac{1}{2}$ the sum of the arcs $BC, CD, DA, AB = \text{right angle}$. $\therefore EG \perp FH$. Again, $APF, BSH, CPE, DSE$ are straight lines since they bisect $\angle s$ $BAC, ABD, BCA, BDA$. Also, $PB, SA$ bisect $\angle s$ $ABC, BAD$. Since $\angle CBD = \angle CAD$ [Th. 3.66(i)], $\therefore \angle CBA - \angle DBA = \angle BAD - \angle BAC$. $\therefore \frac{1}{2}(\angle CBA - \angle DBA) = \frac{1}{2}(\angle BAD - \angle BAC)$ or $\angle PBA - \angle SBA = \angle BAS - \angle BAP$. Hence $\angle PBS = \angle PAS$. $\therefore$ fig. $BPSA$ is cyclic (Th. 3.73). $\therefore \angle BAP = \angle BSP$. $\therefore \angle BAP$ or $\angle BAF = \angle BHF$, $\therefore \angle BSP = \angle BHF$. $\therefore PS \parallel FH$. Similarly, $QR \parallel FH$ and $PQ, RS \parallel GE$. $\therefore$ fig. $PQRS$ is a rectangle whose sides $\parallel EG, FH$, which are perpendicular to one another.

3.12. Construct a triangle $ABC$ having given the vertices of the three equilateral triangles described on its sides outside the triangle.
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CONSTRUCTION: Let $D$, $E$, $F$ be the given vertices. On $DE$, $EF$, $FD$ describe equilateral $\triangle$s $DEG$, $EFH$, $FDJ$. Then $FG$, $DH$, $EJ$ will intersect in one point $L$ and subtend $120^\circ \angle$s to each other. Bisect $FE$ in $K$. Draw $KM$ to make a $60^\circ$ angle with $FD$ cutting $ELJ$ in $A$. Join $AF$ and make $\angle AFC = 60^\circ$ meeting $DLH$ in $C$. Similarly, join $CE$ and make $\angle ECB = 60^\circ$ meeting $GLF$ in $B$. Therefore, $ABC$ is the required $\triangle$. Draw $FN \parallel ELJ$ to meet $MAK$ produced in $N$ (Fig. 80).

**Figure 80**

Proof: \[ \angle FDJ = \angle EDG = 60^\circ, \text{adding } \angle FDE \text{ gives } \angle EDJ = \angle GDF. \] \[ \therefore \triangle EDJ, GDF \text{ are congruent. } \therefore \angle DJE = \angle DFG. \] \[ \therefore \text{quadrilateral } DJFL \text{ is cyclic. } \therefore \angle DLF = 120^\circ. \text{ Hence } EJ, FG \text{ and similarly } DH \text{ meet in one point at } 120^\circ \angle s. \text{ But } \angle KAF = \angle KMF + \angle MFA = \angle MFC \left( \because \angle KMF = \angle AFC = 60^\circ \right). \] Since, also, quadrilateral $ALCF$ is cyclic, \[ \therefore \angle LCF + \angle LAF = 2 \text{ right angles. But } \angle LAF + \angle AFN = 2 \text{ right angles } (FN \parallel AE). \] \[ \therefore \angle LCF = \angle AFN. \] \[ \therefore \angle CAF = \angle CLF = 60^\circ, \therefore \triangle AFC \text{ is equilateral. } \therefore \triangle DCF, NFA \text{ are congruent. } \therefore DC = FN = AE (AFNE \text{ is a } \square). \] Again, quadrilateral $LBEC$ is cyclic. \[ \therefore \angle LCB = \angle LEB \text{ and } \angle CLE = \angle CBE = 60^\circ. \therefore \triangle CBE \text{ is equilateral. } \]\[ \triangle ABE, DBC \text{ are congruent } \left( \because DC = AE \right). \therefore \angle ABE = \angle DBC. \therefore \angle CBE = \angle DBA = 60^\circ. \therefore \text{Quadrilateral } DBLA \text{ is cyclic, } \therefore \angle BDA + \angle BLA = 2 \text{ right angles since } \angle BLA = 120^\circ. \therefore \angle BDA = 60^\circ \text{ also. Hence } \triangle DBA \text{ is equilateral.} \]

3.13. Construct a triangle having given its nine-point circle, orthocenter, and the difference between the base angles. How many solutions are possible?
Analysis: Assume $ABC$ is the required triangle and that $O_1$ is the center of its nine-point circle, $O$ being the orthocenter. Let $G$ be the center of the circumcircle. Then $O_1$ is the middle point of $OG$ [Th. 3.77(i)]. $F$, $H$, $J$ are the mid-points of $CB$, $BA$, $CA$ and $GF$, $GH$, $GJ$ are perpendicular to these sides of the triangle (Fig. 81). Since $ABDE$ is cyclic, \[ \angle DAE = \angle DBE. \]

Also, $AHGJ$ is cyclic. \[ \therefore \angle GAH = \angle GJH. \] Since $H, J$ are the mid-points of $AB, AC$, \[ \therefore \angle GJH = \angle DBE \] (between $||s$). \[ \therefore \angle DAE = \angle GAH. \] Since $\angle C - \angle B = \angle BAD - \angle CAD = \text{given angle}$, \[ \therefore \angle GAO = \text{given angle}. \] Also the radius of the nine-point circle $O_1O = \text{half that of circumcircle} O$ (Th. 3.76, Cor.). Therefore,

Synthesis: Join $OO_1$ and produce it to $G$, so that $OO_1 = O_1G$. On $OG$ as a chord, draw a circle to subtend at the arc $OG$ an angle $\angle GAO = \text{given difference between base angles of } \triangle$. With $G$ as center and radius $= \text{given diameter of } O_1O$, draw a circle cutting $O_1GAO$ in $A$. Join $AO$ and produce it to meet the nine-point circle in $D$. Draw $BDC \perp AD$ to meet $OG$ in $B, C$; then $ABC$ is the required triangle. Proof is obvious. If $OG$ cuts $O_1CO$ in two points the problem will have two solutions. In case $OG$ touches $O$ on $GAO$, the problem will then have only one solution. What would be the condition, if any, for which there will be no solution?

3.14. $O$ is the orthocenter of a triangle $ABC$, $O_1$ is the center of its circumscribed circle, and $D, E, F$ are the centers of the circles drawn to circumscribe triangles $BOC, COA, AOB$. Show that (i) $O_1$ is the orthocenter of triangle $DEF$; (ii) $O$ is the circumcenter of triangle $DEF$; (iii) $A, B, C$ are the circumcenters of triangles $EO_1F, FO_1D, DO_1E$; (iv) all mentioned eight triangles have the same nine-point circle.
Construction: Let $AP$, $BQ$, $CR$ be the perpendiculars from $A$, $B$, $C$ to corresponding sides of $\triangle ABC$. Let also $FE$, $FD$, $DE$ cut $AO$, $BO$, $CO$ at right angles in $X$, $Y$, $Z$. Join $DO_1$, $EO_1$, $FO_1$ to meet the opposite sides of $\triangle DEF$ in $L$, $M$, $N$ respectively; then join $BF$, $BO$, and $BD$ (Fig. 82).

Proof: (i) $\therefore D$, $E$ are two circumcenters of $\triangle s BOC, COA$, $\therefore DE$ bisects the common chord $CO$ and $\perp$ to it. $\therefore AB \perp COR$, $\therefore AB \parallel DE$. Similarly, $EF \parallel BC$ and $FD \parallel AC$. $\therefore AB$ is the common chord in the $\circ s FBA$, $ABC$, $\therefore$ likewise, $FO_1 \perp AB$. $\therefore FO_1 \perp DE$. Similarly, $DO_1$, $EO_1$ when produced are $\perp EF$, $FD$. Hence $O_1$ is the orthocenter of $\triangle DEF$.

(ii) $\therefore \angle BOC$ is supplementary to $\angle BAC$ and $BC$ is a common base, $\therefore \angle BAC = \text{any } \angle$ on $\circ BOC$ on other side of $BC$. $\therefore \angle BDG = \angle BO_1G$. Hence $\circ s BOC$, $ABC$ are equal. Similarly, $\circ s COA$, $BOA$ are each $\circ ABC$. $\therefore BD = BO_1 = BF = CE$. $\therefore DF \perp BO$, $\therefore OD = OF$ (from congruence of $\triangle s BDO$, $BFO$). Similarly, $OD = OE$. Hence $O$ is the circumcenter of $\triangle DEF$.

(iii) Since $\circ s ABC$, $AOB$, $BOC$, $COA$, $DEF$ are equal, $\therefore$ their radii are equal. $\therefore BD = BO_1 = BF$. $\therefore B$ is the circumcenter of $\triangle FOD$. Similarly, $A$, $C$ are circumcenters of $\triangle s EO_1F$, $DO_1E$, and these eight circles are equal.

(iv) $\therefore$ Nine-point $\circ$ of $\triangle ABC$ passes through mid-points of $AO$, $BO$, $CO$, i.e., $X$, $Y$, $Z$, which are also the mid-points of $EF$, $FD$, $DE$,
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\( \therefore \circ PQR, \) the nine-point \( \bigcirc \) of \( \triangle ABC, \) is also the nine-point \( \bigcirc \) of \( \triangle DEF. \) Suppose \( D'O_1 \) cuts \( BC \) in \( G. \) \( \therefore \) \( G \) is the mid-point of \( BC. \) Since \( \circ PQR \) passes through \( Y, G, Z, \) which are the mid-points of the sides of \( \triangle BOC, \) \( \therefore \) \( \circ PQR \) is the nine-point \( \bigcirc \) of \( \triangle BOC \) and similarly of \( \triangle COA, \) \( \triangle AOB. \) Also, \( \circ PQR \) is the nine-point \( \bigcirc \) of \( \triangle EO_1F, \) \( F'O_1D, \) \( D'O_1E \) since it passes through the mid-points of their sides. Therefore, these eight equal circles have the same nine-point \( \circ PQR. \)

3.15. Describe a circle which will pass through a given point \( P \) and touch a given straight line \( MN \) and also a given circle \( ABC. \)

**Analysis:** Suppose \( PP'D \) is the required \( \bigcirc. \) Through centers \( O, \) \( O' \) draw \( AOBE, \) \( O'D \perp MN. \) \( \therefore \) \( O'D \parallel AB, \therefore \) straight line joining \( AD \) passes through \( F \) the point of contact of \( \bigcirc \)s. Join \( BF, \) \( AP. \) Let \( AP \) meet \( \circ PFD \) again in \( P' \) (Fig. 83). \( \therefore \angle BFD, \) \( BED \) are right, hence

![Figure 83](image_url)

\( B, E, D, F \) are concyclic. \( \therefore AB \cdot AE = AF \cdot AD = AP \cdot AP'. \therefore P' \) is a known point. Therefore, the problem is reduced to a simpler one—to describe a \( \bigcirc \) passing through \( P \) and \( P' \) and touching \( MN. \) Hence,

**Synthesis:** Find \( O \) the center of given \( \bigcirc ABC. \) Draw \( AOBE \perp MN. \) In \( AP, \) produced if necessary, find \( P' \) such that the rectangle \( AP \cdot AP' = \) rectangle \( AB \cdot AE. \) This is done by describing \( \bigcirc EB \) and producing \( AP \) to meet its circumference in \( P'. \) Produce \( APP' \) to meet \( MN \) in \( R. \) Draw on \( AP \) the distance \( PP'' = RP'. \) On \( RP'' \) as diameter describe a semi-circle, then draw \( PQ \perp AP'. \) Take \( RD = PQ; \) then \( \circ PP'D \) touches \( MN. \) Since \( PQ^2 = PP'' \cdot PR = RP' \cdot RP = RD^2, \) \( MN \) touches \( \circ PP'D. \) Join \( AD \) and let \( AD \) cut the \( \circ PP'D \) in \( F. \) Now, \( AD \cdot AF = AP \cdot AP' = AB \cdot AE. \) \( \therefore \) \( BEDF \) is cyclic (Th. 3.79). \( \therefore \) \( \angle AFB = \angle BED = \) right angle. \( \therefore \) \( F \) is on \( \bigcirc ABC. \) \( \therefore \) \( \angle O'FD = \angle O'DF = \) alternate \( \angle OAF = \angle OFA, \therefore \) \( OFO' \) is a straight line. \( \therefore \) \( \circ PP'D \) touches \( \bigcirc ABC \) at \( F. \)
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Note: 1. If line $AP$ intersects $MN$, then two circles can be described through $P, P'$ touching $MN$. Also, if the points $A, B$ are interchanged, $P'$ will occupy a different position and two more circles will be obtained. In this case the contact will be internal. Hence the problem has four solutions.

2. This is one of a group of related problems known collectively as the problems of Apollonius. The problem is to construct a circle either passing through one or more given points, tangent to one or more given lines or tangent to one or more given circles in various combinations. This problem would be PLC, meaning that the required circle must pass through a given point $P$, be tangent to a given line $L$, and be tangent to a given circle $C$. We have previously considered in different terminology the more elementary cases PPP and LLL.

3.16. $ABCD$ is a parallelogram. A straight line is drawn through $A$ and meets $CB, CD$ produced in $E, F$. Show that $\frac{AC^2}{AE} = \frac{AE}{AF}$.

**Construction:** Describe circumscribing circle $CFE$. Produce $CA$ to meet circle in $G$. Join $GE, GF$. Draw $DP, BQ$ to make with $AC \angle DCP$, $CBQ = \angle FGC, EGC$ respectively (Fig. 84).

**Figure 84**

**Proof:** \[ \angle CBQ = \angle EGC, \therefore \text{ quadrilateral } BECQ \text{ is cyclic.} \therefore \angle BEG + \angle BQG = 2 \text{ right angles. But } \angle BEG = \angle FCG + \angle FGC = \angle FCG + \angle CDP. \therefore \text{ Supplementary } \angle CPD = \text{ supplementary } \angle BQC. \therefore BQ \parallel DP. \therefore ABCD \text{ is a } \square, \therefore AP = CQ. \text{ Since } BEGQ \text{ is cyclic, } \therefore CB \cdot CE = CQ \cdot CG. \text{ Also, } DFGP \text{ is cyclic. } \therefore CD \cdot CF = CP \cdot CG \text{ [Th. 3.78(i)]. Adding yields } CB \cdot CE + CD \cdot CF = CG(CQ + CP) = CG \cdot AC = AC^2 + AC \cdot AG. \therefore CB \cdot CE + CD \cdot CF = AC^2 + AE \cdot AF. \]
3.17. ABC is any triangle. If AD, AE, AF are the median, bisector of the vertex angle, and altitude respectively and AG is the external bisector of the vertex angle, show that (i) \( 4(AB - AC)^2 \); (ii) \( 4(AB + AC)^2 \).

**Construction:** Draw \( BR, CP \perp AE, AG \) or produced respectively. \( BR, CP \) produced meet \( AC, BA \) in \( S, Q \). Then join \( RD, RF, PD, PF \). (Fig. 85).

**Figure 85**

**Proof:** (i) From congruence of \( \triangle ABR, ASR, BR + RS, \) and \( AB = AS \). Hence \( DR \) is \( \parallel \) and \( = \frac{1}{2} CS \) or \( DR = \frac{1}{2} (AB - AC) \). Since \( DR \) is \( \parallel ACS \), \( \therefore \angle DRA = \angle CAE = \angle BAE \). \( \therefore \) Quadrilateral \( ABRF \) is cyclic, \( \therefore \angle BAE = \angle BFR \). \( \therefore \angle DRA = \angle BFR \). \( \therefore \) \( DR \) touches \( \odot ERF \) (Th. 3.72). \( \therefore \) \( DR^2 = DE \cdot DF \). \( \therefore \) \( 4DE \cdot DF = (AB - AC)^2 \).

(ii) Similarly, from congruence of \( \triangle APC, APQ, PC = PQ, AC = AQ, \) and \( \angle ACP = \angle AQP \). Hence \( DP \) is \( \parallel \) and \( = \frac{1}{2} BQ = \frac{1}{2} (AB + AC) \). \( \therefore \angle AQP = \angle CPD \). \( \therefore \) Quadrilateral \( AFCP \) is cyclic, \( \therefore \angle ACP = \angle AFP \). \( \therefore \angle CPD = \angle AFP \). Adding a right angle gives \( \angle GPD = \angle DFP = \angle FGP + \angle FPG \), eliminating common \( \angle FPG \). \( \therefore \angle DPF = \angle FGP \). \( \therefore \) \( DP \) touches \( \odot FGP \). \( \therefore \) \( 4DP^2 = 4DG \cdot DF = (AB + AC)^2 \).

3.18. \( AB \) is a chord at right angles to a diameter \( CD \) of a given circle, and \( AB \) is nearer \( C \) than \( D \). Draw through \( C \) a chord \( CQ \) cutting \( AB \) in \( P \) so that \( PQ \) is a given length.
ANALYSIS: Suppose \( CPQ \) is the required chord, \( PQ \) being a given length. Join \( DQ \), and bisect \( PQ \) in \( R \) (Fig. 86). Now, \( \angle CQD \) = right angle, \( \therefore \) quadrilateral \( PNDQ \) is cyclic. \( \therefore CP \cdot CQ = CN \cdot CD \). Analyzing, \((CR - PR)(CR + PR) = CN^2 + CN \cdot ND\). \( \therefore CR^2 - PR^2 = CN^2 + CO^2 - NO^2 \). Hence \( CR^2 = CN^2 + CO^2 + PR^2 - NO^2 \), which is known.

Also, \( CP = (CR - PR) \) is known. Therefore,

SYNTHESIS: Draw from \( C \) the line

\[
CP = \sqrt{CN^2 + CO^2 + PR^2 - NO^2 - PR}
\]

This is done geometrically outside the figure and can be easily followed. Then produce \( CP \) until it cuts the circumference in \( Q \). Hence \( CPQ \) is the required chord. The construction of \( CR \) can also be shown on the figure.

3.19. \( D, E \) are the points of contact of the inscribed and escribed circles of any triangle \( ABC \) with the side \( AB \). Show that \( AD \cdot DB = AE \cdot EB \) = the rectangle of the radii of the two circles.

CONSTRUCTION: Let \( O, O_1 \) be the centers of the inscribed and escribed circles touching side \( AB \) of the \( \triangle ABC \). Join \( AO, OB, AO_1, O_1B \) and draw a circle through \( A, O, B, O_1 \). Produce \( OD \) to meet this circle in \( F \) and join \( FO_1 \) (Fig. 87).

Proof: \( \therefore AO, AO_1 \) are the bisectors of \( \angle A \) and \( BO, BO_1 \) are the bisectors of \( \angle B \), \( \therefore \angle OAO_1 = \angle OBO_1 \) = right angle. Hence quadrilateral \( AOBO_1 \) is cyclic, and \( OO_1 \) is a diameter of this circle. \( \therefore \angle OFO_1 \) = right angle and \( EDFO_1 \) is \( \square \). \( \therefore EO_1 = DF. \therefore AD \cdot DB = OD \cdot DF = OD \cdot EO_1 \). Similarly, by producing \( O_1E \) to meet \( OO_1AO \) at \( F_1 \) etc., we have \( AE \cdot EB = OD \cdot EO_1 \).
3.20. AB is a diameter of a circle and CDC' a chord perpendicular to it. A circle is inscribed in the figure bounded by AD, DC and the arc AC and it touches AB at E. Show that BE = BC, and hence give a construction for inscribing this circle.

Construction: Draw diameter POQ \perp AB. Let M be the center of inscribed \( \odot \) touching \( \odot O \) in K. Draw NME \perp AB and QR \perp NME produced. Let also \( \odot M \) touch CDC' in F. Join MF (Fig. 88). The analysis is carried out for the smaller arc, but could just as well be extended to the larger arc BC.
Proof: \( \therefore \) POQ, NME are diameters in the \( \bigcirc \)s O, M which touch one another at K. \( \therefore \) PNK, QEK are straight lines (\( \therefore \) \( \bigcirc \)s M, O touch at K). \( \angle NKE = \text{right angle} = \angle QRE \). \( \therefore \) Quadrilateral KNRQ is cyclic. \( \therefore \) NE·ER = KE·EQ. Hence NE·OQ = AE·EB. \( \therefore \) MEDF is a square. \( \therefore \) NE = 2 DE. \( \therefore \) DE·PQ = AE·EB. \( \therefore \) DE·AB = AE·EB. Hence AE·DB = AE·EB − AE·ED = DE·AB − AE·DE. \( \therefore \) AE·DB = DE·EB = DE\(^2\) + DE·DB. Adding (DE·DB + DB\(^2\)) to both sides gives AB·DB = BE\(^2\). Since BC\(^2\) = AB·DB, \( \therefore \) BE = BC. To construct the inscribed \( \bigcirc \)M, take on AB the distance BE = BC. Draw EM \( \perp \) AB and = ED; then with M as center and ME as radius draw the inscribed \( \bigcirc \) required. An illustration of the inscribed \( \bigcirc \)M\(_1\) on opposite side of CDC\(_1\) is also given.

3.21. Construct a triangle having given the base, the median which bisects the base, and the difference of the base angles.

Analysis: Suppose ABC is the required \( \triangle \). AD is the given median bisecting the given base BC. Produce AD to meet the circumscribing \( \bigcirc \)O in E. Draw AF \parallel BC and join DO and produce it to meet AF in G. Join OA, OF, EF, EB, EC (Fig. 89). Now, since D is the mid-point of BC and AF is \( \parallel \) BC, \( \therefore \) DOG is \( \perp \) AF. \( \therefore \) \( \angle AOG = \angle GOF \). But \( \angle AOF = 2 \angle AEF \) (Th. 3.65). \( \therefore \) \( \angle AOG = \angle AEF \), since \( \angle AEF = \angle AEC - \angle FEC = \angle B - \angle AEB = \angle B - \angle C = \text{given} \). Hence \( \angle AOG = \text{given} \) (\( \angle B - \angle C \)). \( \therefore \) \( \angle AOD = 2 \) right angles \( \angle B - \angle C \). Also, \( 4 \) AD·DE = 4 BD\(^2\) = BC\(^2\). \( \therefore \) DE = BC\(^2\)/4 AD = given. Hence,

Synthesis: Draw the median AD to the given length. Produce it to E such that \( DE = BC^2/4 \) AD. On AD as a chord, describe an arc of
a circle subtending an angle = 2 right angles = \( \angle B - \angle C \). Bisect \( AE \) in \( L \) and draw \( LO \perp AD \) to meet this arc in \( O \). Join \( OD \) and draw \( BDC \perp \) to it, cutting the circle with center \( O \) and radius \( OA \) in \( B, C \). ABC is then the required \( \triangle \).

3.22. \( AB, AC \) are drawn tangents to a circle with center \( O \) from any point \( A \). \( AB, AC \) are bisected in \( D, E \). If any point \( F \) is taken on \( DE \) or produced, show that \( AF \) is equal to the tangent from \( F \) to the circle. Prove also that if a straight line is drawn from \( F \) to cut the circle in \( G, H \), then \( AF \) will be tangent to the circle described on the triangle \( AGH \).

**Construction:** Let \( FK \) be the tangent from \( F \) to the circle. Join \( OB, OF, OD, OA, OK, CB \) (Fig. 90).

**Proof:** Let \( OA \) cut \( ED \) in \( L \). Since \( AB = AC \), it is easily shown that \( OA \) is the \( \perp \) bisector of \( BC \). \( \therefore \) \( D, E \) are the mid-points of \( AB, AC, \therefore \). \( DE \) is \( \parallel BC \) and hence \( \perp AO \), since \( \angle OBA \) = right angle (Th. 3.68). Therefore, \( DO^2 = DB^2 + BO^2 \). But \( DO^2 = DL^2 + LO^2 \). Also, \( DB^2 = AD^2 = AL^2 + DL^2 \). \( \therefore \) \( LO^2 = AL^2 + OK^2 \) (\( BO = OK \) or \( LO^2 = AL^2 = OK^2 \)). Hence \( FO^2 - AF^2 = OK^2 \). \( \therefore \) \( FO^2 - OK^2 = AF^2 \). Since \( FK^2 = FO^2 - OK^2 \) (in right-angled triangle \( FKO \), \( \therefore \) \( AF = FK \). \( \therefore \) \( FK^2 = FG \cdot FH \) [Th. 3.78(ii)], \( \therefore \) \( AF^2 = FG \cdot FH \). Therefore, \( AF \) is tangent to the circle described on \( \triangle AGH \) (Th. 3.80).

3.23. \( ABCD \) is a quadrilateral drawn inside a circle. If \( BA, CD \) are
produced to meet in $E$ and $AD$, $BC$ in $F$, prove that the circle with $EF$ as diameter cuts the first circle orthogonally.

**Construction:** Let $O$ be the center of the $\bigcirc ABCD$ and $M$ be the center of the $\bigcirc$ with diameter $EF$ which cuts the first one in $N$. Make $AG$ meet $EF$ in $G$, so that $\angle AGE = \angle ABC$. Join $OB$, $ON$, $OE$, $OM$, $OF$, $MN$ (Fig. 91).

**Proof:** Quadrilateral $ABFG$ is cyclic (Th. 3.75). \( \therefore \ EA \cdot EB = EG \cdot EF \) [Th. 3.78(i)]. Since $\angle ABC = \angle ADE$, \( \therefore \ \angle AGE = \angle ADE \). Hence quadrilateral $ADGE$ is cyclic also. \( \therefore \ FD \cdot FA = FG \cdot FE \). Adding gives $EA \cdot EB + FD \cdot FA = EG \cdot EF + FG \cdot EF = EF^2$. \( \therefore \ OE^2 = \text{rad.}^2 \quad \bigcirc O + EA \cdot EB \) and $OF^2 = \text{rad.}^2 \quad \bigcirc O + FD \cdot FA$, \( \therefore \ OE^2 + OF^2 = 2 \ OB^2 + EF^2 = 2 \ OB^2 + 4 \ ME^2 = 2 \ OM^2 + 2 \ ME^2$. Therefore, $OM^2 = OB^2 + ME^2 = ON^2 + MN^2$. \( \therefore \ \angle ONM \) is right (Th. 2.52). \( \therefore \ ON, MN \) are radii of both $\bigcirc$s, \( \therefore \) each is tangent to the other $\bigcirc$.

Hence $\bigcirc$ on $EF$ as diameter cuts $\bigcirc ABCD$ orthogonally.

3.24. From the middle point $C$ of an arc $AB$ of a circle, a diameter $CD$ is drawn and also a chord $CE$ which meets the straight line $AB$ in $F$. If a circle, drawn with center $C$ to bisect $FE$, meets $BD$ in $G$, prove that $EF = 2 \ BG$.

**Construction:** Suppose $\bigcirc$ with center $C$ bisects $FE$ in $K$. Join $BE$, $BC$, $GC$ (Fig. 92).

**Proof:** Since $CD$ is a diameter in the $\bigcirc CBD$, \( \therefore \ \angle DBC = \) right angle. Hence in $\triangle GBC$, $CG^2 = BG^2 + BC^2$. But $CG = CK = \text{radii in } \bigcirc C$. \( \therefore \ CK^2 = BG^2 + BC^2 \) (1). \( \therefore \ CK^2 = CE^2 +EK^2 +$
2. $CE \cdot EK$ and $CE^2 + 2CE \cdot EK = CE(CE + 2EK) = CE \cdot CF$, \therefore $CK^2 = EK^2 + CE \cdot CF$ (2).

Since $\angle BFE$ is measured by half the difference of the arcs $AC$, $BE$ or $BC$, $BE$, i.e., $CE$, \therefore $\angle BFE = \angle CBE$. \therefore $CB$ touches $\odot$ on $\angle BFE$ [Th. 3.72(ii)]. \therefore $CB^2 = CE \cdot CF$.

Hence (2) becomes $CK^2 = EK^2 + CB^2$ (3). From (1) and (3), \therefore $BG = EK$ or $EF = 2BG$.

3.25. $ABC$ is a triangle drawn in a circle, having $AC$ greater than $AB$. $DE$ is a diameter in the circle drawn at a right angle to the base $BC$, so that $A, D$ are at the same side of $BC$. From $A$ a perpendicular $AF$ is drawn to $DE$. If $G$ is the point of intersection of $DE$ and $BC$, show that (i) $4(AB + AC)^2$; (ii) $4(AC - AB)^2$.

**Construction:** Draw $EP, EQ \perp s AC, AB$ or produced is necessary. Join $AE, BE, CE, FP, PG, CD$ (Fig. 93).

**Proof:** (i) \therefore $DE$ is a diameter $\perp BC$, \therefore $D, E$ are the mid-points of the larger and smaller arcs $BC$. \therefore $AE$ bisects $\angle A$. Hence $\triangle s AEP, AEQ$ are congruent \therefore $AP = AQ$ and $EP = EQ$. Similarly, $\triangle s ECP, EBQ$ are congruent. \therefore $CP = BQ$. \therefore $2AP = (AB + AC)$ and $2CP = (AC - AB)$. \therefore $AF, EP$ are $\perp s DE, AC$ respectively, \therefore $\triangle s$ quadrilateral $AFPE$ is cyclic. \therefore $\angle EAP = \angle EFP$. But $\angle EAP$ or $\angle EAC = \angle EDC = \angle ECG$. Since $ECPG$ is cyclic $\triangle s$, \therefore $\angle ECG = \angle EPG$. Hence $\angle EFP = \angle EPG$. \therefore $EP$ touches $\odot PGF$. \therefore $EP^2 = EG \cdot EF$. Since $AP^2 = AE^2 - EP^2 = ED \cdot EF - EG \cdot EF = DG \cdot EF$, \therefore $4DG \cdot EF = (AB + AC)^2$.

(ii) Similarly, $CP^2 = CE^2 - EP^2 = EG \cdot ED - EG \cdot EF = EG \cdot DF$. \therefore $4DF \cdot GE = (AC - AB)^2$. 
3.26. Show that in any triangle, the perpendiculars from the vertices to the opposite sides of its pedal triangle are concurrent and that the area of the triangle is equal to the product of the radius of the circumscribed circle and half the perimeter of the pedal triangle.

Construction: ABC is a triangle drawn inside a circle with center O. AD, BE, CF are the $\perp$ s to the opposite sides and DEF its pedal triangle. Join AO, BO, CO cutting EF, FD, DE in G, K, L respectively. Produce AO to meet $\odot$ in N, then join CN, OD, OE, OF (Fig. 94).
Proof: \( \triangle ACN \) and \( \triangle CFB \) have \( \angle C = \angle F \) right and \( \angle ACN = \angle CBF \) or \( \angle CBA \). \( \therefore \angle CAN = \angle CBF \). But, since \( BFEC \) is a cyclic \( \triangle \), \( \therefore \angle BCF = \angle BEF = \angle CAN \) or \( \angle EAG \). \( \therefore \angle BEG + \angle GEA = \angle EAG = \angle CAN = \angle BCF = \angle BEF \). \( \therefore \angle EAG + \angle GEA = \angle BCF = \angle BEF \). \( \therefore \angle AG \perp EF \) and passes through \( O \). Similarly, \( BK, CL \) are \( \perp \)'s \( DF, DE \) and also pass through \( O \). Hence \( AG, BK, CL \) are concurrent at the center \( O \). In the quadrilateral \( AFOE \), the diagonals \( AO, FE \) are at right angles. \( \therefore \) quadrilateral \( AFOE = \frac{1}{2} AO \cdot FE \). Similarly, quadrilateral \( FBDO = \frac{1}{2} BO \cdot FD \) and quadrilateral \( EODC = \frac{1}{2} CO \cdot ED \). Hence \( \triangle ABC = \) quadrilaterals \( AFOE + FBDO + EODC = \frac{1}{2} \) radius \( AO(\text{FE + FD + ED}) \). \( = \text{radius } O \times \text{half the perimeter of pedal } \triangle DEF \).

3.27. Construct a right-angled triangle having given the hypotenuse and the bisector of either one of the base angles.

Analysis: Suppose \( \triangle ABC \) is the required \( \triangle \) right-angled at \( A \). Draw \( \bigcirc \) with center \( O \) to circumscribe \( \triangle ABC \) and bisect \( \angle C \) by \( CD \), which when produced meets \( \bigcirc \) in \( E \) and the tangent at \( B \) in \( F \) (Fig. 95). \( \therefore \angle FBE = \angle BCE = \angle ECA = \angle EBA. \therefore \angle BEC = \) right

\[ \text{Figure 95} \]

angle, i.e., \( BE \perp DF \). \( \therefore \) \( \triangle BED \), \( \triangle BEF \) are congruent. \( \therefore \angle DE = EF \). Draw \( OG \perp CB \) to meet \( CD \) in \( G \). Since \( OG \parallel BF \) (both \( \perp CB \)), \( \therefore G \) is the mid-point of \( CF \). Hence \( CD = CG + GD = GF + GD = 2GE \), which is given. Therefore,

Synthesis: Bisect \( BO \) in \( O_1 \) and describe a circle center \( O_1 \) and radius equal to \( \sqrt{O_1^2 - (CD/4)^2} \), which is known, since \( BC, CD \)
are given. From $C$ draw a tangent $CK$ to $\odot O_1$ to cut $\odot$ on $OB$ as diameter in $H, J$. With $C$ as center, and radii $CH, CJ$, draw the arcs $HG, JE$ to meet $\perp$ from $O$ to $BC$ in $G$, and $CG$ produced in $E$ respectively (i.e., rotating $HJ$ about $C$ to positive $GE$). Therefore, $HK = CD/4$, i.e., $\frac{1}{4}$ of the given base angle bisector. $\therefore CH \cdot CJ = CG \cdot CE = CO \cdot CB$, $\therefore$ quadrilateral $EBOG$ is cyclic (Th. 3.79). $\therefore \angle BEG = \angle GOC = \text{right}$. Hence $E$ lies on the $\odot O$ with $BC$ as diameter. Draw $\angle BCG = \angle ECA$. Therefore, $ABC$ is the required $\triangle$.

3.28. $D$ is any point on the diameter $AB$ of a circle, and $DC$ is drawn perpendicular to $AB$ to meet the circle in $C$. A semi-circle is described on $BD$ as diameter on the same side as $ABC$. A circle with diameter $ME$ is described to touch $DC$ in $E$ and the two circles in $F, G$. Show that $AB \cdot ME = CD^2$ and hence derive a method of describing the circle $MEF$.

CONSTRUCTION: Let $O, O_1$ be the centers of $\odot$s with $BD, ME$ as diameters. Draw their common tangent at $G$ to meet $DC$ in $K$. Join $O_0, OK, O_1K, FM, MB, FE, EA$. Draw $AH \perp ME$ produced and produce $CD$ to meet $\odot$ in $L$ (Fig. 96).

**Figure 96**

**Proof:** $KG$ is $\perp$ line of centers $OO_1$ at $G$. Also, $KO, K_1$ bisect $\angle DKG, EKG$ (since $KE, KG, KD$ are tangents to $\odot$s $O_1, O$). $\therefore OK$ is $\perp KO_1$. $\therefore KG^2 = OG \cdot O_1G$. $\therefore EK = KG = KD$, $\therefore DE^2 = 4KG^2 = 4OG \cdot O_1G = DB \cdot ME$ (1).

Now, since $\odot$s $ME, AB$ touch at $F$ and $ME, AB$ are diameters. $\therefore FMB, FEA$ are straight lines. $\therefore CE \cdot EL = FE \cdot EA$. $\therefore \angle MFE$
= \angle AHE = \text{right}, \therefore \text{quadrilateral } FMAH \text{ is cyclic.} \therefore FE \cdot EA = ME \cdot EH. \text{ But } ADEH \text{ is a rectangle.} \therefore AD = HE. \therefore CE \cdot EL = ME \cdot AD. \therefore ME \cdot AD = CE^2 + 2CE \cdot ED \quad (2). \text{ Adding (1) and (2) yields } ME \cdot DB + ME \cdot AD = DE^2 + CE^2 + 2CE \cdot DE \text{ or } AB \cdot ME = CD^2. \text{ Therefore, } ME = CD^2/AB \text{ and } DE^2 = DB \cdot ME = (DB \cdot CD^2/AB). \text{ Accordingly, in order to describe } \odot O_1, \text{ take a distance } DE \text{ on } DC \text{ equal to } \sqrt{(DB \cdot CD^2/AB)}, \text{ which is known. Draw } EM \perp CD \text{ and equal to } CD^2/AB, \text{ which is also known. Hence } \odot \text{ on } EM \text{ as diameter is the required } \odot. 

3.29. ABC, A'B'C' \text{ are equilateral triangles inscribed in two concentric circles } ABCP, A'B'C'P'. \text{ Prove that the sum of the squares on } PA', PB', PC' \text{ is equal to the sum of the squares on } P'A, P'B, P'C.

\text{CONSTRUCTION: Join } C'O \text{ and produce it to meet } A'B' \text{ in } E; \text{ then join } OB', PE, PO, P'O (Fig. 97).

\textbf{Proof: } \therefore A'B'C' \text{ is an equilateral } \triangle, \therefore C'O \text{E is } \perp \text{ bisector of } A'B'. \text{ In } \triangle PA'B', PA'^2 + PB'^2 = 2PE^2 = 2EB'^2. \text{ Adding } PC'^2 \text{ to both sides, } \therefore PA'^2 + PB'^2 + PC'^2 = 2PE^2 + 2EB'^2 + PC'^2. \therefore OE = \frac{1}{2} OC', \text{ since } O \text{ is the centroid of } \triangle A'B'C', ABC, \text{ it is easily proved that } PC'^2 + 2PE^2 = 3PQ^2 + 6OE^2. \therefore PA'^2 + PB'^2 + PC'^2 = 2EB'^2 + 3PQ^2 + 6OE^2 = 2(EB'^2 + OE^2) + 3PQ^2 + 4OE^2 = 2OB'^2 + OC'^2 + 3PQ^2 = 3(PO'^2 + PQ^2). \text{ Similarly, } P'A^2 + P'B^2 + P'C^2 = 3(PO'^2 + PO^2).

3.30. Four circles are described to touch the sides of a triangle. Show that the square on the distance between the centers of any two circles with the square
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on the distance between the centers of the other two circles is equal to the square on the diameter of the circle through the centers of any three.

**Construction:** Let $M, D, E, F$ be the centers of the inscribed and escribed circles opposite $A, B, C$ respectively. Join $AMD, CMF$ and the sides of $\triangle DEF$ and let $EN$ be a diameter in $\circ DEF$. Join also $ND, NF$ (Fig. 98).

![Figure 98](image)

**Proof:** $AMD, CMF$ are straight lines bisecting $\angle A, C$. Also, $EAF, FBD, DCE$ are straight lines bisecting $\angle A, B, C$ externally. Hence $AMD, CMF$ are bisectors of $\angle A, B, C$. $	herefore M$ is the orthocenter of $\triangle DEF$. $	herefore EN$ is a diameter in $\circ DEF$, $	herefore \angle NDE$ = right angle. $	herefore ND$ is $\parallel CF$. Hence $FNDM$ is a parallelogram. $	herefore DN = FM$. Hence $EN^2 = ED^2 + DN^2 = ED^2 + FM^2$ and, similarly, $EN^2 = EF^2 + MD^2$ (see Problem 3.14).

3.31. $AD, BE, CF$ are perpendiculars drawn from the vertices of a triangle $ABC$ on any diameter in its inscribed circle. Show that the perpendiculars $DP$, $EQ$, $FR$ on $BC$, $CA$, $AB$ respectively are concurrent.

**Construction:** Join $AE, AF, BD, BF, CD, CE$ (Fig. 99).

**Proof:** In $\triangle DBC$, $DP \perp BC$. $	herefore BP^2 - PC^2 = DB^2 - DC^2 = BE^2 + ED^2 - DF^2 - FC^2$. Similarly, $CQ^2 - QA^2 = EC^2 - EA^2 = EF^2 + FC^2 - ED^2 - AD^2$ and $AR^2 - RB^2 = AF^2 - FB^2 = AD^2 + DF^2 - EF^2 - BE^2$.

Adding yields $(BP^2 - PC^2) + (CQ^2 - QA^2) + (AR^2 - RB^2) = 0$. Therefore, $DP, EQ, FR$ are concurrent (see Problem 2.20).
3.32. D is a fixed point inside a given circle with center O. AB is a chord in the circle which always subtends a right angle at D. Show that the rectangle contained by the two perpendiculars OC, DE on AB is constant and that the sum of the squares of the perpendiculars on AB from two other fixed points, which can be determined, is also constant.

Construction: Join OD and bisect it in K. Draw KL ⊥ AB and PKQ || AB meeting OC, DE or produced in P, Q. Join CD, CK, EK, OA. (Fig. 100).
Proof: (i) \[OC \cdot DE = DE (EQ + DQ) = DE^2 + 2 DE \cdot DQ = EK^2 - DK^2.\] Since \(\angle ADB = \text{right angle,} \ \therefore CD = \frac{1}{2} AB = AC\) (Th. 1.27). \(\therefore CD^2 + CO^2 = 2 CK^2 + 2 DK^2 = AC^2 + CO^2 = AO^2\) = constant, \(\therefore DK\) is fixed (since \(OD\) is of a fixed length), \(\therefore CK = EK\) is fixed. Therefore, \((EK^2 - DK^2)\) or \(OC \cdot DE\) is constant.

(ii) On \(OD\) as diagonal, construct a square \(OGDF\); then \(F, G\) are also fixed points. Draw \(FM, GN \perp AB\) and join \(FKG\). \(\therefore FM^2 + GN^2 = 2 ((FM + GN)/2)^2 + 2 ((FM-GN)/2)^2 = 2 KL^2 + 2 KQ^2 = 2 EQ^2 + 2 KQ^2 = 2 EK^2.\) Since \(EK\) is fixed, \(\therefore (FM^2 + GN^2)\) is constant.

Miscellaneous Exercises

1. If the perpendiculars from \(A, B, C\) on the opposite sides of the triangle \(ABC\) meet the circumscribed circle in \(G, H, K\), show that the area of the hexagon \(AHCGBK\) is twice that of the triangle \(ABC\).

2. \(P\) is the orthocenter of the triangle \(ABC, D\) any point in \(BC\). If a circle be described with center \(D\) and radius \(DP\) meeting \(AP\) produced in \(E\), then \(E\) lies on the circumscribing circle of the triangle.

3. \(AD\) the bisector of the angle \(BAC\) cuts the base \(BC\) in \(D\), and \(BH\) a parallel to \(AD\) meets \(CA\) produced in \(H\). Prove that the circles circumscribing the triangles \(BAC, HDC\) cut \(AD\) produced in points equidistant from \(A\).

4. \(AB\) is a fixed chord of a circle, \(AP, BQ\) any two chords parallel to each other. Prove that \(PQ\) touches a fixed circle.

5. A straight line \(LPM\) meets the lines \(CX, CY\) in \(L, M\). At \(M\) make the angle \(YMD\) equal to the angle \(CPM\). At \(C\) make the angle \(YCD\) equal to the angle \(PCL\) and let \(MD, CD\) meet in \(D\). Prove that the angles \(DLP, PCL\) are equal.

6. The bisectors of the angles of the triangle \(ABC\) inscribed in a circle intersect in \(O\) and being produced meet the circle in \(D, E, F\). Prove that \(O\) is the orthocenter of the triangle \(DEF\).

7. \(P\) is any point on the circumference of a circle which passes through the center \(C\) of another circle. \(PQ, PR\) are tangents drawn from \(P\) to the other circle. Show that \(CP, QR\) meet on the common chord of the circles.

8. The angle \(A\) of the triangle \(ABC\) is a right angle. \(D\) is the foot of the perpendicular from \(A\) on \(BC\) and \(DM, DN\) are drawn perpendicular to \(AB, AC\) respectively. Show that the angles \(BMC, BNC\) are equal.

9. Through the point of contact of two given circles which touch each other, either externally or internally, draw a straight line terminated by the circles which shall be equal to a given straight line.

10. If two triangles equal in every respect be placed so as entirely to coincide, and one be turned in its plane about one angular point, show that
the line joining that angular point to the point of intersection of the opposite sides will bisect the angle between those sides.

11. From a point $P$ inside a triangle $ABC$ perpendiculars $PD$, $PE$, $PF$ are drawn to $BC$, $CA$, $AB$ respectively. If the angle $EDF$ is equal to $A$, prove that the locus of $P$ is an arc of the circle passing through $B$, $C$ and the center of the circle circumscribing $ABC$.

12. With any point $G$ on the circumference of a circle as center, a circle is described cutting the former in $B$, $C$. From a point $H$ on the second circle as center, a circle is described touching $BC$. Prove that the other tangents from $B$, $C$ to the third circle intersect on the circumference of the first circle.

13. A triangle $ABC$ is described in a circle. From a point $P$ on the circumference perpendiculars are drawn to the sides, meeting the circle again in $A'$, $B'$, $C'$ respectively. Show that $AA'$, $BB'$, $CC'$ are parallel.

14. From the vertices $B$, $C$ of the triangle $ABC$ perpendiculars $BE$, $CF$ are drawn to the opposite sides meeting them in $E$, $F$. Show that the tangents at $E$, $F$ to the circle through $A$, $E$, $F$ intersect in $BC$.

15. $ABC$ is a triangle, $Q$ any point on the circumscribing circle, $QM$ the perpendicular from $Q$ on $AB$. If $CQ$ meets $AB$ in $L$, and if the diameter through $C$ meet the pedal of $Q$ in $N$, prove that $C$, $L$, $M$, $N$ lie on a circle.

16. The lines joining the points where the bisectors of the angles between the opposite sides of a quadrilateral inscribed in a circle meet the sides form a rhombus.

17. $A$, $B$, $C$ are any points on the circumference of a circle. $D$ is the middle point of the arc $AB$, $E$ the middle point of the arc $AC$. If the chord $DE$ cuts the chords $AB$, $AC$ in $F$, $G$ respectively, prove that $AF = AG$.

18. $ABC$ is an arc of a circle whose center is $O$. $B$ is the middle point of the arc, and the whole arc is less than a semi-circumference. From $P$, any point in the arc, $PM$, $PN$, $PQ$ are drawn perpendicular respectively to $OA$, $OB$, $OC$ or produced if necessary and $NR$ is drawn perpendicular to $OA$. Show that $PM$ and $PQ$ are together double of $NR$.

19. $O$ is the center of the circle inscribed in the triangle $ABC$. Straight lines are drawn bisecting $AO$, $BO$, $CO$ at right angles. Show that these straight lines intersect on the circle $ABC$.

20. $A$, $B$, $C$ lie on a circle. Through the center, lines are drawn parallel to $CA$, $CB$ meeting the tangents at $A$, $B$ in $D$, $E$ respectively. Prove that $DE$ touches the circle.

21. $ABC$ is a circle whose center is $O$. Any circle is described passing through $O$ and cutting the circle $ABC$ in $A$, $B$. From any point $P$ on the circumference of this second circle, straight lines are drawn to $A$, $B$ and, produced if necessary, meet the first circle in $A'$, $B'$ respectively. Prove that $AB'$ is parallel to $A'B$. 
22. If the chords which bisect two angles of a triangle inscribed in a circle be equal, prove that either the angles are equal, or the third angle is equal to the angle of an equilateral triangle.

23. $ABC$ is a triangle inscribed in a circle. From $D$ the middle point of one of the arcs subtended by $BC$, perpendiculars are drawn to $AB$, $AC$. Prove that the sum of the distances of the feet of these perpendiculars from $A$ is equal to the sum or difference of the sides $AB$, $AC$ according as $A$ and $D$ are on opposite sides or on the same side of $BC$.

24. $AB$ is a diameter of a circle, $CD$ a chord perpendicular to $AB$, $DP$ any other chord meeting $AB$ in $Q$. Prove that $CA$ and also $CB$ make equal angles with $CP$, $CQ$.

25. $O$ is a point on the circumference of the circle circumscribing the triangle $ABC$. Prove that if the perpendiculars dropped from $O$ on $BC$, $CA$, $AB$ respectively meet the circle again in $a$, $b$, $c$, the triangle $abc$ is equal in all respects to $ABC$.

26. Construct a square such that two of its sides shall pass through two points $B$, $C$ respectively and the remaining two intersect in a given point $A$.

27. From any point $P$ perpendiculars $PA'$, $PB'$, $PC'$ are drawn to the sides $BC$, $CA$, $AB$ of a triangle, and circles are described about the triangles $PA'B'$, $PB'C'$, $PC'A'$. Show that the area of the triangle formed by joining the centers of these circles is $\frac{1}{4}$ of the area of the triangle $ABC$.

28. Draw a straight line cutting two concentric circles such that the chord of the outer circle will be twice that of the inner.

29. Draw through $B$, one of two fixed points $A$, $B$, a line which will cut the circle on $AB$ as diameter in $C$ and the perpendicular from $A$ to $AB$ in $D$ so that $BC$ will be equal to $AD$.

30. $ABC$, $A'B'C'$ are two triangles equiangular to each other inscribed in two concentric circles. Show that the straight lines $AA'$, $BB'$, $CC'$ form a triangle equiangular with the triangle formed by joining the feet of the perpendiculars from the vertices of $ABC$ on the opposite sides.

31. Two circles touch each other externally at $C$ and a straight line in $A$ and $B$. $AC$, $BC$ produced meet the circles again in $E$, $F$ respectively. Show that the square on $EF$ is less than the sum of the squares on the diameters of the circles by the rectangle contained by the diameters.

32. Two circles, lying wholly outside one another, are touched by four common tangents. Show that if $AB$ be an outer common tangent, the two inner common tangents meet $AB$ in points $P$, $Q$, such that $AP$ is equal to $BQ$.

33. Given two parallel straight lines and a point between them, draw a straight line parallel to a given straight line, the part of which intercepted between the parallels will subtend a given angle at the given point.
34. In any triangle $ABC$ if the internal and external bisectors of the angle $A$ meet the opposite side in $I, K$ respectively and if $M$ be the middle point of $IK$, the triangles $ACM$, $BAM$ are equiangular to each other and $MA$ touches the circle $ABC$.

35. The circumference of one circle passes through the center of a second circle and the circles intersect in $A, B$. Prove that any two chords through $A, B$ of the second circle which intersect on the circumference of the first circle are equal.

36. A circle revolves around a fixed point in its circumference. Show that the points of contact of tangents to the circle which are parallel to a fixed line lie on one or other of two fixed circles.

37. Describe a circle with given radius to pass through a given point and touch a given circle.

38. Of all triangles on the same base and having the same vertical angle, the isosceles has the greatest perimeter.

39. Two equal circles $EAB, FAB$ intersect in $A, B$. $BE$ is drawn touching the circle $BAF$ at $B$ and meeting the circle $BAE$ in $E$. $EA$ is joined and produced to meet the circle $BAF$ in $F$. A line $BC'MG$ is drawn through $B$ at right angles to $EAF$, meeting it in $M$, the circle $BAE$ in $C$, and the circle $BAF$ in $C'$. $C'A$ is joined and produced to meet $BE$ in $K$. Prove that $KM$ is parallel to $BF$.

40. $A$ is a fixed point on a circle. From any point $B$ on the circle, $BD$ is drawn perpendicular to the diameter through $A$. Prove that the circle through $A$ touching the chord $BD$ at $B$ is of constant magnitude.

41. $ABC$ is a triangle and $AL, BM, CN$, its perpendiculars, meet the circumscribing circle in $A', B', C'$. $S$ is any point on the circle. Show that $SA', SB', SC'$ meet $BC, CA, AB$ in points in a straight line which passes through the orthocenter of the triangle $ABC$.

42. From a point on a circle three chords are drawn. Prove that the circles described on these as diameters will intersect in three points in a straight line.

43. The alternate angles of any polygon of an even number of sides inscribed in a circle are together equal to a number of right angles less by two than the number of sides of the polygon.

44. Given a point $P$ either inside or outside a given circle. Show how to draw through $P$ straight lines $PA, PB$ cutting the circle in $A, B$ containing a given angle so that the circle circumscribing the triangle $PAB$ will pass through the center of the given circle.

45. $A', B', C'$ are the vertices of equilateral triangles described externally on the sides of a triangle $ABC$. Prove that $AA', BB', CC'$ are equal, that they are equally inclined to each other, and that they are concurrent.

46. The sides of a triangle are cut by a circle concentric with the inscribed circle, and each vertex of the hexagon formed by the intersections is
joined to the opposite vertex. Show that the triangle so formed is equiangular to the triangle formed by the points of contact of the inscribed circle with the sides.

47. Two equal circles touch at A. A circle of twice the radius is described having internal contact with one of them at B and cutting the other in P, Q. Prove that the straight line AB will pass through P or Q.

48. From an external point P tangents are drawn to a circle whose center is C and CP is joined. If the points of contact be joined with the ends of the diameter perpendicular to CP, prove that the points of intersection of the joining lines and the points of contact are equally distant from P.

49. Describe a triangle equiangular to a given triangle whose sides pass through three given points and whose area shall be a maximum.

50. Show how to draw a pair of equal circles on two parallel sides of a parallelogram as chords, so as to touch each other, and show that the circles so drawn on the two pairs of parallel sides intersect at angles equal to those of the parallelogram.

51. E is the intersection of the diagonals of a quadrilateral inscribed in a circle. FEG is the chord which is bisected in E. Prove that the part of this chord intercepted between the opposite sides of the quadrilateral is also bisected in E.

52. (a) P is the orthocenter of the triangle ABC inscribed in a circle whose center is O. If the parallelogram BACG be completed, show that G is a point on the circumference of the circle which passes through B, P, C, and hence prove that AP is twice the perpendicular drawn from O to BC.

(b) ABCD is a quadrilateral inscribed in a circle. Prove that the orthocenters of the triangles ABC, BCD, CDA, DAB lie on an equal circle.

53. Two segments of circles on the same straight line and on the same side of it, which contain supplementary angles, intercept equal lengths on perpendiculars to their common chord.

54. Prove that if two adjacent sides of a square pass through two fixed points, the diagonal also passes through a fixed point. Hence show how to describe a square about a given quadrilateral.

55. A', B', C' are the feet of the perpendiculars from the vertices of a triangle ABC on the opposite sides BC, CA, AB, D, E, F the middle points of those sides respectively. O is the orthocenter and G, H, K the middle points of AO, BO, CO. Show that GD, HE, FK are equal and concurrent.

56. Through a fixed point which is equidistant from two parallel straight lines, a straight line is drawn terminated by the two fixed straight lines and on it as base is described an equilateral triangle. Prove that the vertex of this triangle will lie on one of two straight lines.
57. Describe a circle to touch a given circle and a given straight line, and to have its center in another given straight line.

58. A straight line $AB$ slides between two fixed parallel straight lines to which it is perpendicular. Find the position of $AB$ when it subtends the greatest possible angle at a fixed point.

59. Construct a triangle given the base, the vertical angle, and the sum of the squares on the sides.

60. If a circle touch a given circle and also touch one of its diameters $AB$ at $C$, prove that the square on the straight line drawn from $C$ at right angles to $AB$ to meet the circumference of the given circle is equal to half the rectangle contained by the diameters of the circles.

61. One diagonal of a quadrilateral inscribed in a circle is bisected by the other. Show that the squares on the lines joining their point of intersection with the middle points of the sides are together half the square on the latter diagonal.

62. A circle passing through the vertex $A$ of an equilateral triangle $ABC$ cuts $AB$, $AC$ produced in $D$, $E$ respectively and $BC$ produced both ways in $F$, $G$. Show that the difference between $AD$ and $AE$ is equal to the difference between $BF$ and $CG$.

63. $A$, $B$ are points outside a given circle. $C$ is a point in $AB$ such that the rectangle $AB$, $AC$ is equal to the square on the tangent from $A$ to the circle. $CD$ is drawn to touch the circle in $D$ and $AD$ is drawn cutting the circle again in $E$. If $BE$ cut the circle again in $F$, show that $DF$ is parallel to $AB$.

64. $CD$ is a chord of a given circle parallel to a given straight line $AB$. $G$ is a point in $AB$ such that the rectangle $AB$, $BG$ is equal to the square on the tangent from $B$ to the circle. If $DG$ cuts the circle in $F$, prove that $AC$, $BF$ intersect on the circle.

65. $AB$ is a diameter of a semi-circle on which $P$ and $Q$ are two points. From $AQ$, the distance $QR$ is cut off equal to $QB$. Prove that if $AR$ is equal to $AP$, the tangent from $Q$ to the circle around $ARP$ is the side of a square of which $BP$ is a diameter.

66. $A$ is the center of a circle. $PN$ is a perpendicular let fall on the radius $AB$ from a point $P$ on the circle. Show that the tangent from $P$ to the circle of which $AB$ is a diameter is equal to the tangent from $Q$ to the circle which of which $AN$ is a diameter.

67. $A$, $B$, $C$, $D$ are four fixed points in a straight line. A circle is described through $A$, $B$ and another through $C$, $D$ to touch the former. Prove that the point of contact lies on a fixed circle.

68. Two chords $AB$, $CD$ of a circle intersect at right angles in a point $O$ either inside or outside the circle. Prove that the squares on $AB$, $CD$ are together less than twice the square on the diameter by four times the square on the line joining $O$ to the center of the circle.
69. From a point $P$ two tangents $PT$, $PT'$ are drawn to a given circle whose center is $O$ and a line $PAB$ cutting the circle in $A$, $B$. If $PO$, $TT'$ intersect in $C$, prove that $TC$ bisects the angle $ACB$.

70. If perpendiculars are drawn from the orthocenter of a triangle $ABC$ to the bisectors of the angle $A$, their feet are collinear with the middle point of $BC$ and the nine-point center.

71. $ABC$ is a right-angled triangle at $A$ inscribed in a circle. $D$ is any point on the smaller arc $AC$, and $DE$ is drawn perpendicular to $BC$ cutting $AC$ in $F$. From $F$ a perpendicular is drawn to $AC$ meeting the circle on $AC$ as diameter in $G$. Show that $DC = CG$.

72. $ABCD$ is a quadrilateral inscribed in a circle and $P$ any point on the circumference. From $P$ perpendiculars $PM$, $PN$, $PQ$, $PR$ are drawn to $AB$, $BC$, $CD$, $DA$ respectively. Show that $PM \cdot PQ = PN \cdot PR$.

73. $ABC$ is a right-angled triangle at $A$. From any point $D$ on the hypotenuse $BC$, a perpendicular $DFG$ is drawn to $BC$ to meet $AB$, $AC$ or produced if necessary in $F$, $G$ respectively. Show that (a) $DF^2 = BD \cdot DC - AF \cdot FB$; (b) $DG^2 = BD \cdot DC + GA \cdot GC$.

74. $ABCD$ is a concyclic quadrilateral. If $BA$, $CD$ are produced to meet in $K$ and also $AD$, $BC$ in $L$, show that the square on $KL$ is equal to the sum of the squares on the tangents from $K$ and $L$ to the circle.

75. $G$ is the center and $AB$ a diameter of the circle $ADEB$, and the chord $DE$ is parallel to $AB$. Join $AD$ and draw $AP$ perpendicular to $AD$ to meet $ED$ produced in $P$ and join $PC$ and $AE$. Show that $PC^2 = AP^2 + AC^2 + AE^2$.

76. $AB$ is a diameter of a circle with center $O$. $CD$ is a chord parallel to $AB$. If $P$ is any point on $AB$ and $Q$ is the mid-point of the smaller arc $CD$, prove that (a) $AP^2 = PB^2 = CP^2 + PD^2$; (b) $CP^2 + PD^2 = 2 PQ^2$.

77. If two circles touch externally, show that the square on their common tangent is equal to the rectangle contained by the diameters.

78. $A$, $B$ are two fixed points on a diameter of a circle with center $C$ such that $CA = CB$. If any chord $DAE$ is drawn through $A$ in the circle, show that the sum of the squares on the sides of the triangle $BDE$ is constant.

79. If two chords of a circle intersect at right angles, show that the sum of the squares on the four segments is equal to the square on the diameter.

80. A circle touches one side $BC$ of a triangle and the other sides $AB$, $AC$ produced, the points of contact being $D$, $F$, $E$. If $I$ be the center of the inscribed circle, prove that the areas of the triangles $IAE$, $IAF$ are together equal to that of the triangle $ABC$.

81. $M$, $N$ are the centers of two intersecting circles in $A$, $B$. From $A$, $CAB$, $DAE$ are drawn at right angles such that $D$, $B$ lie on the circumference.
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of the circle \( M \) and \( C, E \) on that of circle \( N \). Prove that \( CB^2 + DE^2 = 4 \, MN^2 \).

82. If from a fixed point \( T \), without a circle whose center is \( O \), \( TA, TB \) are drawn equally inclined to \( TO \) to meet the concave and convex arcs respectively in \( A \) and \( B \), show that \( AB, TO \) meet in a fixed point.

83. If the inscribed circle in a right-angled triangle at \( A \) touches the hypotenuse \( BC \) at \( D \), then \( DB \cdot DC = \) the area of triangle \( ABC \).

84. Prove that if three circles intersect, their three common chords are concurrent.

85. \( OA, OB \) are tangents to a given circle whose center is \( C \) and \( CO \) cuts \( AB \) in \( D \). Prove that any circle through \( O, D \) cuts the given circle orthogonally.

86. Through a point on the smaller of two concentric circles, draw a line bounded by the circumference of the larger circle and divided into three equal parts at the points of section of the smaller circle.

87. Describe a circle which will pass through two given points and cut a given circle orthogonally.

88. If the square on the line joining two points \( P, Q \) be equal to the sum of the squares on the tangents from \( P, Q \) to a circle, then \( Q \) is on the straight line joining the points of contact of tangents drawn to the circle from \( P \).

89. \( OA, OB \) are straight lines touching a circle in \( A, B \). \( OC \) is drawn perpendicular to \( AB \) and bisected in \( D \). \( DF \) is drawn touching the circle in \( F \). Prove that \( CFO \) is a right angle.

90. If \( P \) be any point in the circumference of a circle described about an equilateral triangle \( ABC \), show that the sum of the squares on \( PA, PB, PC \) is constant.

91. If \( O \) be any point in the circumference of the circle inscribed in an equilateral triangle \( ABC \), prove that the sum of the squares on \( OA, OB, OC \) is constant.

92. The tangents at \( A, B \), the ends of a chord \( AB \) of a circle, whose center is \( C \), intersect in \( E \). Prove that the tangents at the ends of all chords of the circle which are bisected by \( AB \) intersect on the circle whose diameter is \( CE \).

93. In a circle the arc \( AB \) is equal to the arc \( BC \). \( P \) is any other point on the circle. From \( B \) let fall \( BQ, BR \) perpendiculars on \( AP, CP \) respectively. Show that \( OQ^2 + OR^2 = 2 \, OP^2 \), where \( O \) is the center of the circle.

94. One circle cuts another at right angles. Show that, if tangents be drawn from any point in one circle to meet the other, then the chord of contact passes through the opposite extremity of the diameter of which the first point is one extremity.

95. The circle circumscribing the triangle \( ABC \) is touched internally at the point \( C \) by a circle, which also touches the side \( AB \) in \( F \) and cuts the
sides $BC$, $AC$ in $D$, $E$. If the tangent at $D$ to the inner circle cuts the outer circle in $G$, $H$, prove that $BH$, $BG$, $BF$ are all equal, and that $CF$ bisects the angle $ACB$.

96. Two circles, whose centers are $A$, $B$, intersect in $C$, $D$. $E$ is the middle point of $AB$. If $F$ be any point in $CD$, then the chords intercepted by the circles on a line through $F$ perpendicular to $EF$ are equal.

97. Describe through two given points a circle such that the chord intercepted by it on a given unlimited straight line may be of given length.

98. $ABC$ is a triangle right-angled at $A$. From $D$, any point in the circumference of the circle described on $BC$ as diameter, a perpendicular is drawn to $BC$ meeting $AB$ in $E$. From $E$ is drawn a perpendicular to $AB$, meeting in $F$ the circle described on $AB$ as diameter. Prove that $BF = BD$.

99. Tangents are drawn to a circle at the ends of a chord $PQ$, and through $O$, a point in $PQ$, a straight line $COAB$ is drawn parallel to one of the tangents, meeting the other in $B$ and the circle in $A$, $C$. Show that if $A$ bisects $OB$, then $O$ will bisect $BC$. Show also how to determine the point $O$ that this may be the case.

100. Draw through a given point $P$ a straight line $PQR$ to meet two given lines in $Q$, $R$ so that the rectangle $PQ$, $PR$ will be equal to a given rectangle.

101. $ABC$ is a triangle, $D$, $E$, $F$ the middle points of its sides. With the orthocenter of the triangle as center any circle is described cutting $EF$, $FD$, $DE$ in $P$, $Q$, $R$ respectively. Prove that $AP = BQ = CR$.

102. $AB$ is a chord of a circle whose center is $O$. $P$ is a point in $AB$. If $OP$ be produced to $Q$ so that the rectangle $OP\cdot OQ$ is equal to the square on the radius of the circle, prove that $QA$, $QB$ make equal angles with $QO$.

103. $C$ is the middle point of $AB$, a chord of a circle whose center is $O$. A point $P$ is taken in the circumference, whose distance $PD$ from $AB$ is equal to $AC$. $M$ is the middle point of $PD$ and $CF$ is drawn parallel to $OM$ to meet $PD$ in $F$. Show that $CF = FP$.

104. $A$, $B$ are two fixed points taken on a diameter of a semi-circle whose center is $C$ such that they are equidistant from $C$. If $AP$, $BQ$ are two parallel lines terminated by the circumference, show that $AP\cdot BQ$ is constant.

105. $D$ is the middle point of a straight line $AB$. If $AM$, $BN$, $DE$ are tangents to any circle, show that $AM^2 + BN^2 = 2(AD^2 + DE^2)$.

106. $P$ is any point outside a circle whose center is $O$. From $P$, $PA$ and $PBOC$ are drawn tangent and transversal through $O$ respectively. Through $P$ and $A$ another circle is drawn tangent to $PBC$ at $P$, and intersects the first circle in $D$. Show that $AD$ produced bisects $PE$, where $E$ is the projection of $A$ on $BC$. 

CIRCLES AND TANGENCY
107. \( O, M \) are the centers of two intersecting circles in \( A, B \). Draw a transversal from \( A \) or \( B \) to the two circles so that the rectangle of the two segments contained in the circles is equal to a given rectangle.

108. \( M, N \) are the centers of two intersecting circles in \( A, B \). From \( A \) a straight line \( CAD \) is drawn and terminated by the circles \( M, N \) in \( C, D \) respectively. With \( C \) and \( D \) as centers, describe two circles to cut separately the circles \( N, M \) orthogonally. Show that these two circles together with the circle on \( CD \) as diameter are concurrent.

109. Describe a triangle \( ABC \) having given the rectangle of the sides \( AB, AC \), the difference of the base angles and the median bisecting the base \( BC \).

110. \( O \) is the center of a circle and \( BC \) is any straight line drawn outside it. \( OD \) is drawn perpendicular to \( BC \); then the distance \( DE \) is taken on \( OD \) equal to the tangent from \( D \) to the circle. If \( F \) is any other point on \( BC \), show that \( FE \) is equal to the tangent from \( F \) to the circle.

111. \( A, B, C, D \) are points lying in this order in a straight line, and \( AC = a \) inches, \( CD = b \) inches, \( BC = 1 \) inch. A perpendicular through \( C \) to the line cuts the circle on \( BD \) as diameter in \( E \). \( EFG \) passes through the center of the circle on \( AC \) as diameter, cutting the circumference in \( F, G \). Show that the area of the rectangle \( EF \cdot EG \) is \( b \) square inches, and also that the number of inches in the length of \( EF \) is a root of the equation \( x^2 + ax - b = 0 \).

112. Two circles are drawn touching the sides \( AB, AC \) of a triangle \( ABC \) at the ends of the base \( BC \) and also passing through \( D \) the middle point of \( BC \). If \( E \) is the other point of intersection of the circles, prove that the rectangle \( DA, DE \) is equal to the square on \( DC \).

113. Construct a triangle given the altitude to the base, the median of the base, and the rectangle of the other two sides.

114. \( A, B \) are fixed points and \( AC, AD \) fixed straight lines such that \( BA \) bisects the angle \( CAD \). If any circle through \( A, B \) cuts off chords \( AK, AL \) from \( AC, AD \), prove that \( (AK + AL) \) is constant.

115. Construct a triangle having given the bisector of the vertical angle, the rectangle of the sides containing this angle, and the difference of the base angles.

116. In a given circle, draw two parallel chords from two given points on its circumference such that their product may be of given value.

117. Draw two intersecting circles having their centers on the same side of the common chord \( AB \), and draw a diameter of the smaller circle. Describe a circle within the area common to the two circles, which will touch them both and have its center in the given diameter. When will there be two circles fulfilling the given conditions?

118. \( O \) is any point in the plane of the triangle \( ABC \). Perpendiculars to \( OA, OB, OC \) drawn through \( A, B, C \) form another triangle \( A'B'C' \). Prove that the perpendiculars from \( A', B', C' \) on \( BC, CA, AB \) respectively meet in a point \( O' \), and that the center of the circle \( ABC \) is the middle point of \( OO' \).
CHAPTER 4
RATIO AND PROPORTION

Theorems and Corollaries

Ratios

4.82. A. (i) If \( a/b > 1 \) and \( x > 0 \), then \( (a + x)/(b + x) < a/b \).
   (ii) If \( 0 < a/b < 1 \) and \( x > 0 \), then \( (a + x)/(b + x) > a/b \).

B. If \( a/b = c/d \), then \( b/a = d/c \) and \( a/c = b/d \).

C. (i) If \( a/b = c/d \), then \( ad = bc \).
   (ii) Conversely, if \( ad = bc \), then \( a/b = c/d \) or \( a/c = b/d \).

D. (i) If \( a/b = b/c \), then \( b^2 = ac \).
   (ii) Conversely, if \( b^2 = ac \), then \( a/b = b/c \).

E. If \( a/b = b/c \), then \( a/c = a^2/b^2 = b^2/c^2 \).

F. If \( a/b = c/d \), then \( (a + b)/b = (c + d)/d, (a - b)/b = (c - d)/d \) and \( (a + b)/(a - b) = (c + d)/(c - d) \).

G. If \( a/b = x/y \) and \( b/c = y/z \), then each of these ratios = \((lx + my + nz)/(la + mb + nc)\), where \( l, m, n \) are any quantities whatever, positive or negative.

Proportion and Similar Polygons

4.83. (i) If a straight line is drawn parallel to one side of a triangle, it divides the other two sides proportionally. (ii) Conversely, if a straight line divides two sides of a triangle proportionally, it is parallel to the third side.

Corollary. If two straight lines \( PT, P'T' \) cut three parallel straight lines \( AB, CD, EF \) at \( Q, R, S; Q', R', S' \) respectively, then \( QR/RS = Q'R'/R'S' \) and \( QS/RS = Q'S'/R'S' \) and, conversely, if \( AB \) is parallel to \( EF \) and \( QR/RS = Q'R'/R'S' \) or \( QS/RS = Q'S'/R'S' \), then \( CD \) is parallel to \( AB \).

4.84. If two triangles are mutually equiangular, their corresponding sides are proportional, and the triangles are similar.

4.85. If the three sides of one triangle are proportional to the three sides of another triangle, the two triangles are mutually equiangular.

4.86. If two triangles have an angle of the one equal to an angle of the other, and the sides about these equal angles proportional, the triangles are mutually equiangular and similar.
4.87. If the straight lines joining a point to the vertices of a given polygon are divided (all internally or all externally) in the same ratio, the points of division are the vertices of a polygon which is mutually similar to the given polygon.

4.88. If $O$ is any point and $ABC \cdots$ is any polygon, and straight lines $AOA', BOB', COC', \cdots$ are drawn such that $AO : A'O = BO : B'O = CO : C'O = \cdots = k$, then the resulting polygon $A'B'C' \cdots$ is said to be homothetic to the original polygon $ABC \cdots$; $O$ is the center of homothecy and $k$ the homothetic ratio. The polygons are similar, and when $k$ equals 1, congruent.

Corollary. If two polygons are similar and similarly placed, the straight line joining any pair of opposite vertices of one figure is parallel to that joining the corresponding opposite vertices of the other.

4.89. If a polygon is divided into triangles by lines joining a given point to its vertices, any similar polygon can be divided into corresponding similar triangles.

4.90. (i) If the vertex angle of a triangle is bisected, internally or externally, by a straight line which cuts the base, or the base produced, it divides the base, internally or externally, in the ratio of the other sides of the triangle. (ii) Conversely, if a straight line through the vertex of a triangle divides the base, internally or externally, in the ratio of the other sides, it bisects the vertex angle, internally or externally.

Areas of Similar Polygons

4.91. The ratio of the areas of two triangles or of two parallelograms of equal or the same altitude is equal to the ratio of their bases.

4.92. The ratio of the areas of two similar triangles, or of two similar polygons, is equal to the ratio of the squares on corresponding sides.

Corollary. The ratio of the areas of two similar triangles is equal to the ratio of (i) the squares of any two corresponding altitudes; (ii) the squares of any two corresponding medians; (iii) the squares of the bisectors of any two corresponding angles.

4.93. The areas of two triangles which have an angle of one equal to an angle of the other are to each other as the products of the sides including the equal angles.

Corollary. The areas of two triangles that have an angle of one supplementary to an angle of the other are to each other as the products of the sides including the supplementary angles.

4.94. If a triangle $ABC$ equals another triangle $A'B'C'$ in area, and the angle $A$ equals angle $A'$, then $AB \cdot AC = A'B' \cdot A'C'$. Conversely, if in two triangles $ABC, A'B'C'$ angles $A, A'$ are equal and $AB \cdot AC = A'B' \cdot A'C'$, then the two triangles are equal.

4.95. If two parallelograms are mutually equiangular, then the ratio of their areas is equal to the product of the ratios of two pairs of corresponding sides.
4.96. If ABCD is a parallelogram and E is any point on the diagonal BD from which FEL and GEK are drawn parallel to AB and BC respectively and terminated by the sides, then the parallelograms BGEF and DKEL are similar to ABCD.

4.97. In any right-angled triangle, any rectilinear figure described on the hypotenuse is equal to the sum of two similar and similarly described figures on the sides containing the right angle.

**Ratio and Proportion in Circles**

4.98. In equal circles, angles either at the centers or at the circumferences have the same ratio to one another as the arcs on which they intercept; so also have the sectors.

**Corollary.** The sectors are to each other as their angles.

4.99. If any angle of a triangle is bisected by a straight line which cuts the base, the rectangle contained by the sides of the triangle is equal to the sum of the rectangle contained by the segments of the base and the square on the line which bisects the angle.

4.100. If from any vertex of a triangle a straight line is drawn perpendicular to the base, the rectangle contained by the sides of the triangle is equal to the rectangle contained by the perpendicular and the diameter of the circle described about the triangle.

4.101. Ptolemy General Theorem. The rectangle contained by the diagonals of a quadrilateral is less than the sum of the rectangles contained by opposite sides unless a circle can be circumscribed about the quadrilateral, in which case it is equal to their sum.

**The Measure of a Circle**

4.102. The circumferences of any two circles have the same ratio as their radii.

**Corollary 1.** The ratio of the circumference of any circle to its diameter is a constant denoted by \( \pi \) and equals approximately 3.14.

**Corollary 2.** If \( C \) is the circumference, \( D \) the diameter, and \( R \) the radius of a circle, then \( C = \pi D = 2\pi R \).

**Corollary 3.** If \( A \) is the number of degrees in an arc of a circle, or in its subtended central angle, then the length of the arc may be expressed by \( (A/360)2\pi R \).

4.103. The area of a circle is equal to half the product of its circumference and radius and expressed by \( \pi R^2 \).

**Corollary 1.** The area of a sector of a circle may be expressed in the form \( (A/360)\pi R^2 \), where \( A \) is the central angle of the sector.

**Corollary 2.** The area of a segment is (i) area of a sector — area of triangle (if central angle \(< 180^\circ\) ); (ii) area of a sector + area of triangle (if central angle \(> 180^\circ\) ).
Corollary 3. The area of any two circles are to each other as the squares of their radii or as the squares of their diameters.

Solved Problems

4.1. Two straight lines \( AD, AE \) are drawn from the vertex \( A \) of a triangle \( ABC \) to make equal angles with \( AB, AC \) and meet the base \( BC \) in \( D, E \). If \( D \) is nearer to \( B \) and \( E \) nearer to \( C \), show that \( AB^2 : AC^2 = BD \cdot BE : CE \cdot CD \).

Construction: On \( \triangle ADE \) draw a circle cutting \( AB, AC \) in \( F, G \). Join \( FG, DG \) (Fig. 101).

![Figure 101](image)

Proof: \( \angle FAD = \angle FGD \) (in \( \odot ADE \)). Also, \( \angle GAE = \angle GDE \).

\( \angle FAD = \angle GAE \) (hypothesis), \( \therefore \) \( \angle FGD = \angle GDE \).

\( FG \) is \( || DE || BC \). Hence in \( \triangle ABC, BF/AB = CG/AC \) (Th. 4.83).

\( BF/CG = AB/AC \). Multiplying yields \( (BF/CG) \cdot (AB/AC) = AB^2/AC^2 \). But since \( BF \cdot AB = BD \cdot BE \) and \( CG \cdot AC = CE \cdot CD \), \( \therefore \) \( AB^2/AC^2 = (BD/CE) \cdot (BE/CD) \).

4.2. \( ABC \) is any triangle. From the vertices three equal straight lines \( AD, BE, CF \) are drawn to meet \( BC, CA, AB \) in \( D, E, F \) respectively. If another three lines are drawn from any point \( M \) inside the triangle parallel to these equal lines and meeting \( BC, CA, AB \) in \( P, Q, R \) respectively, show that \( AD = MP + MQ + MR \).

Construction: Draw from \( M \) the lines \( GMH, MJ, MK || BC, AC, AB \) (Fig. 102).

Proof: In similar \( \triangle MKP, ABD, MP/AD = MK/AB \). Also in similar \( \triangle MKJ, ABC, MK/AB = KJ/BC \) (Th. 4.84). \( \therefore \) \( KJ/BC = MP/AD \) (1). Similarly, in similar \( \triangle MQH, BEC, MQ/BE = MH/BC \). Since \( MJCH \) is a \( \square \), \( \therefore \) \( MH = CJ \). \( \therefore \) \( MQ/BE = CJ/BC \) (2). Again, in similar \( \triangle MGR, CBF, MR/CF = MG/BC \). Since \( MGBK \) is another \( \square \), \( \therefore \) \( MG = BK \). \( \therefore \) \( MR/CF = BK/BC \) (3).
Since $AD$, $BE$, $CF$ are equal, adding ratios (1), (2), (3) gives

$\frac{MP + MQ + MR}{AD} = \frac{KJ + CJ + BK}{BC} = \frac{BC}{BC}$.

$\therefore AD = MP + MQ + MR$.

**Corollary.** If from $M$ any point inside an equilateral $\triangle$ perpendiculars are drawn to the three sides, then the sum of the three perpendiculars is always equal to any altitude in the triangle (see Exercise 1.46).

4.3. $ABCD$ is a quadrilateral and a transversal line is drawn to cut $AB$, $AD$, $CD$, $BC$, $AC$, $BD$ or produced in $E$, $F$, $G$, $H$, $I$, $J$. Show that $\frac{EF}{GH} = \frac{FI}{GI} \cdot \frac{EJ}{HJ}$.

**Construction:** From $D$ draw the line $LDMN \parallel HF$ to meet $BC$, $BA$, $CA$ produced in $L$, $M$, $N$ (Fig. 103).
Proof: In similar \( \triangle F A I, D A N, F I/D N = A F/AD \). Also, in similar \( \triangle A F E, A D M, E F/D M = A F/AD \). \( \therefore \) \( E F/D M = F I/D N \) \( (1) \). In \( \triangle B L M \), \( E H \parallel LM \) and \( BD \) is a transversal. Hence \( D M/D L = E J/H J \) \( (2) \). Similarly, in \( \triangle C L N \), \( H I \parallel LN \) and \( CD \) is a transversal. \( \therefore \) \( D L/G H = D N/G I \) \( (3) \). Multiplying \( (1), (2), (3) \) gives \( (E F/D M) \cdot (D M/D L) \cdot (D L/G H) = (F I/D N) \cdot (E J/H J) \cdot (D N/G I) \). \( \therefore \) \( E F/G H = (F I/H J) \cdot (E J/G I) \).

4.4. Bisect a triangle \( ABC \) by (i) a straight line perpendicular to its base \( BC \); (ii) a straight line parallel to one side; (iii) a straight line parallel to a given direction.

Construction: (i) Bisect \( BC \) in \( M \). Draw \( M N \perp CB \) to meet \( AC \) in \( N \). \( N P \) is drawn \( \perp CA \) cutting the circumference of \( \odot \) on \( CA \) as diameter in \( P \). Join \( CP \) and take \( CE \) on \( CA = CP \). Then perpendicular \( ED \) to \( BC \) is the required bisector. Draw \( AF \perp BC \) and join \( AM \) (Fig. 104(i)).

\[ \text{(i) Figure 104 (ii)} \]

\( \text{Proof:} \ \triangle ACM = \triangle ACF = CM/CF \) (Th. 4.91). Since \( CM/CF = CN/CA = CN \cdot CA/CA^2 = CP^2/CA^2 = CE^2/CA^2 \), \( \therefore \) \( \triangle ACM = \triangle ACF = CE^2/CA^2 \). But \( \triangle CED = \triangle ACF = CE^2/CA^2 \), since \( DE \parallel AF \) (Th. 4.92). \( \therefore \) \( \triangle CED = \triangle ACM = \frac{1}{2} \triangle ABC \).

Construction: (ii) Bisect the base \( BC \) in \( M \). Then draw \( MF \perp BC \) to meet the circle on \( BC \) as diameter in \( F \). Join \( CF \) and take \( CD \) on \( BC \) equal to \( CF \). Draw \( DE \parallel AB \) meeting \( AC \) in \( E \). \( DE \) is the required bisector. Join \( AM \) [Fig. 104(ii)].
Proof: Since $M$ is the mid-point of $BC$, \[ \triangle ACM = \frac{1}{2} \triangle ABC. \]
Since $DE$ is $\parallel AB$, \[ \triangle CDE = \frac{1}{2} \triangle ABC \] (Th. 4.92) \[ = \frac{CF^2}{CB^2} = \frac{CM \cdot CB}{CB^2} = \frac{CM}{CB} = \frac{1}{2}. \therefore \triangle CDE = \frac{1}{2} \triangle ABC. \]

(iii) This is a more general case of (i) and is always possible for some given direction. Since in Fig. 104(i), if from $A$ a parallel is drawn to the given direction to cut $BC$ in a point $F$, a similar construction of $DE$ $\parallel$ to the line $AF$ can be established as in case (i) by making $MN$ $\parallel$ $AF$ also. From some other given direction, for example, $\parallel BC$, it may be necessary to use a median other than $AM$.

4.5. From a given point $P$ in the base $BC$ produced of a triangle $ABC$, draw a straight line to cut the sides of the triangle, so that if lines be drawn parallel to each side from the point where it intersects the other, they shall meet on the base $BC$.

Analysis: Suppose $PEF$ be the required line, so that $D$ the point of intersection of $ED$ drawn $\parallel AB$ and $FD$ drawn $\parallel AC$, is in $BC$ (Fig. 105). \[ \therefore FB \parallel DE, \therefore PB/PD = PF/PE. \therefore FD \parallel EC, \therefore PD/PC \]

\[ = PF/PE. \therefore PB/PD = PD/PC. \] Hence $PD$ is the mean proportional between $PB$ and $PC$; i.e., $PD^2 = PB \cdot PC$.

Synthesis: On $PB$ as diameter, describe a semi-circle. Draw $CR \perp BP$ to meet the circle in $R$. With $P$ as center draw $PD = PR$; hence $PD^2 = PR^2 = PB \cdot PC$ or $PD$ is the mean proportional between $PB, PC$. Then draw $DE \parallel AB$. Join $PE$ and produce it to meet $AB$ in $F$. Join $DF$. $DF$ will be $\parallel AC$, \[ \therefore PB/PD = PD/PC \] and $PB/PD$.
\[ \frac{PF}{PE}. \therefore DE \parallel AB, \therefore \frac{PD}{PC} = \frac{PF}{PE}. \therefore \frac{DC}{PC} = \frac{FE}{PE}. \therefore FD \parallel AC. \]

4.6. \(A, B, C, D\) are any four points on a straight line. On \(AB, CD\) as diameters, two circles are described and a common tangent \(EF\) is drawn to touch them in \(E, F\) respectively and meet \(AD\) produced in \(G\). Show that

(i) \(AC \cdot BG = BD \cdot AG\); (ii) \(EF^2 = AC \cdot BD\).

Construction: Let \(M, N\) be the centers of the two circles on \(AB, CD\) as diameters. Join \(AE, BE, ME, CF, DF, NF\) (Fig. 106).

![Figure 106](image)

Proof: (i) Since \(ME \parallel NF\) (\(\perp EF\), Th. 3.68), \(\angle AME = \angle CNF\).

\(\angle AEH = \frac{1}{2} \angle AME\) and \(\angle CFH = \frac{1}{2} \angle CNF\) (Th. 3.72), \(\therefore \angle AEH = \angle CFH\). \(\therefore AE \parallel CF\). Similarly, \(BE \parallel DF\). In \(\triangle AEG\), \(\frac{AG}{AC} = \frac{EG}{EF}\). Also, in \(\triangle BEG\), \(\frac{BG}{BD} = \frac{EG}{EF}\). \(\therefore \frac{AG}{AC} = \frac{BG}{BD}\). \(\therefore \frac{AC}{BG} = \frac{BD}{AG}\).

(ii) In \(\triangle AEG\), \(\frac{AG}{EF} = \frac{CG}{FG}\). Also, in \(\triangle BEG\), \(\frac{EF}{BD} = \frac{FG}{GD}\). Since \(GF^2 = GD \cdot GC\) or \(\frac{GC}{FG} = \frac{FG}{GD}\), \(\therefore \frac{AC}{EF} = \frac{EF^2}{AC \cdot BD}\). \(\therefore \frac{EF^2}{AC \cdot BD}\).

4.7. \(P\) and \(Q\) are two points in the sides \(AB, CD\) respectively of a quadrilateral \(ABCD\) such that \(AP : PB = CQ : QD\). Prove that if \(QA, QB, PC, PD\) be joined, the sum of the areas of the triangles \(QAB, PCD\) is equal to the area of the quadrilateral \(ABCD\).

Construction: Join \(AC, PQ\) (Fig. 107).

Proof: Assume \(\frac{AP}{PB} = \frac{CQ}{QD} = \frac{a}{b}\). \(\therefore \triangle APQ \parallel \triangle QAB = \frac{a(a + b)}{a}\) (Th. 4.91). \(\therefore \triangle QAB = \frac{\triangle APQ}{(a + b)/a}\). Similarly, \(\triangle CPQ \parallel \triangle PCD = \frac{a(a + b)}{a}\). \(\therefore \triangle PCD = \frac{\triangle CPQ}{(a + b)/a}\). Hence, by adding, \(\triangle QAB + \triangle PCD = \frac{\triangle APQ + \triangle CPQ}{(a + b)/a}\). But \(\triangle ACQ \parallel \triangle ACD = \frac{a(a + b)}{a}\). \(\therefore \triangle ACD = \frac{\triangle ACQ}{(a + b)/a}\). Also, \(\triangle ABC = \frac{\triangle ACP}{(a + b)/a}\). \(\therefore \triangle ACD + \triangle ABC = \frac{\triangle ACQ}{(a + b)/a}\).
4.8. A point $P$ is given in the base of a triangle. Show how to draw a straight line cutting the sides of the triangle and parallel to the base, which will subtend a right angle at $P$.

**Construction:** Let $ABC$ be the given triangle and $P$ is in $BC$. Bisect $BC$ in $D$. With center $D$ and radius $DB$ describe a $\odot BDC$. Produce $AP$ to meet the circle in $E$ and draw $PF \parallel DE$ meeting $AD$ in $F$. Then draw $GFH \parallel BC$ meeting $AB$, $AC$ in $G$, $H$ respectively. Join $GP$, $HP$ (Fig. 108).

**Proof:** In $\triangle ABD$, $AF/FG = AD/DB$. Also, in $\triangle ADE$, $AF/FP = AD/DE$ (Th. 4.83). Since $DB = DE$ (radii in $\odot D$), $\therefore AF/FG = AE/FP$. $\therefore FG = FP$. Similarly, $FH = FP$. $\therefore FH = FH$ (since $AD$ is a median and $GFH$ is $\parallel BC$), $\therefore FG = FH = FP$. $\therefore F$ is the center of the circle $GPH$ of which $GH$ is a diameter. Hence $\angle GPH$ is a right angle.
4.9. AB is a straight line and C is any point on it. On AB, BC, AC three equilateral triangles ABD, BCE, ACF are drawn such that the two small triangles are on the opposite side of the large one. If M, N, L are the centers of the inscribed circles in these triangles respectively, show that MN = ML.

**Construction:** Join AM, BM, BN, CN, CD, MD (Fig. 109).

![Figure 109](image)

**Proof:** Since MB, NB bisect $\angle$s ABD, ABE respectively and $\therefore$ each $= 60^\circ$, $\therefore \angle MBD = \angle NBC$. Similarly, $\angle NCB = \angle MDB$.

$\therefore \triangle MBD, NBC$ are mutually equiangular and hence similar.

$\therefore MB/BD = NB/BC$. Again, $\angle CBD = \angle NBM = 60^\circ$. Hence $\triangle$s CBD, NBM are similar (Th. 4.86).

$\therefore CD/BD = MN/BM$. Similarly, $ML/AM = CD/AD$. Since $BD = AD$, $\therefore MN/BM = ML/AM$.

$\therefore BM = AM$ also, $\therefore MN = ML$.

4.10. Describe an equilateral triangle which will be equal in area to a given triangle ABC.

**Construction:** Describe on BC an equilateral triangle BCG. Draw AE $\parallel$ BC meeting CG in E, and EH $\perp$ CG to meet the semi-circle described on CG as diameter in H. Take CD = CH on CG and draw DF $\parallel$ BG. $\triangle$CDF is the required equilateral $\triangle$ (Fig. 110).

**Proof:** $CH^2 = CD^2 = CE \cdot CG$ (Th. 3.81). In $\triangle$BCG, DF $\parallel$ BG, $\therefore \triangle CDF/\triangle CBG = CD^2/CG^2 = CE \cdot CG/CG^2 = CE/CG$ (Th. 4.92).

But $\triangle CEB/\triangle CBG = CE/CG = \triangle ABC/\triangle CBG$ (Th. 4.91) (since
\( \triangle CEB = \triangle ABC \) between the \( \parallel \)s \( AE, BC \). Hence \( \triangle CDF = \triangle ABC \) and it is an equilateral \( \triangle \) (because \( DF \parallel BG \)). Therefore, \( \triangle CDF \) is the required \( \triangle \).

**Figure 110**

4.11. \( AB, AC \) are two radii in a circle with center \( A \) making an angle \( 120^\circ \). Show how to inscribe a circle inside the sector \( ABC \) and, if \( D \) is the middle point of the arc \( BC \) of the sector, find the ratio of the area of this inscribed circle to that of the other circle inscribed in the segment \( BDC \).

**Construction:** Bisect \( \angle BAC \) by \( AD \). Draw \( DE \perp AB \); then bisect \( \angle ADE \) by \( DF \) meeting \( AB \) in \( F \). \( FM \) is drawn \( \perp AB \) meeting \( AD \) in \( M \). Then \( M \) is the center of the inscribed \( \bigcirc \) in the sector \( ABC \) whose radius is \( MF \) or \( MD \). Let \( AD, BC \) intersect in \( G \) and bisect \( DG \) in \( N \). Describe another \( \bigcirc \) with center \( N \) and radius \( ND \) inscribed in the segment \( BDC \) (Fig. 111).

**Figure 111**

**Proof:** \( \therefore MF \parallel DE \), \( \therefore \angle MFD = \angle FDE \). But \( \angle FDE = \angle FDA \) (since \( DF \) bisects \( \angle ADE \)) \( \therefore \angle MFD = \angle FDA \). \( \therefore MF = MD \). \( \therefore MF \perp AB \) and \( D \) is the middle point of the arc \( BC \), \( \therefore \bigcirc \) with \( M \) as center and \( MF = MD \) as radii touches \( AB, arc \ BC \), and similarly \( AC \). In \( \triangle AMF, \angle FAM = 60^\circ \) and \( \angle F = \text{right angle} \). Hence \( MF \)
\[ AM = 2/\sqrt{3} MF. \quad AD = AM + MD = (2/\sqrt{3}) \]
\[ MF + MF = ((2 + \sqrt{3})/\sqrt{3}) MF. \quad \triangle OMA = MF^2 : AD^2 \quad (\text{Th. 4.103, Cor. 3}) = MF^2 : ((2 + \sqrt{3})/\sqrt{3})^2 MF^2 = 1 : 4.64 \quad (\text{approx.}) \quad (1). \]

Again, in \( \triangle GBA \), \( \angle GAB = 60^\circ \) and \( AG \perp CB \). \( \therefore AG = \frac{1}{2} AB = GD. \quad \therefore \triangle OHA = ND^2 : AD^2 = 1 : 16 \quad (2). \) Hence, dividing (1) by (2) yields \( OQ : ON = 16 : 4.64 = 3.45 : 1 \) (approx.).

4.12. \( ABCD \) is a quadrilateral in a circle whose diagonals intersect at right angles. Through \( O \) the center of the circle \( GOG' \), \( HOH' \) are drawn parallel to \( AC, BD \) respectively, meeting \( AB, CD \) in \( G, H \) and \( DC, AB \), produced in \( G', H' \). Show that \( GH, G'H' \) are parallel to \( BC, AD \) respectively.

**Construction:** Let \( OG, OH \) meet \( BD, AC \) in \( E, F \) respectively. Produce \( BA, CD \) to meet in \( K \). Draw \( KLM \perp CA, OG \) meeting them in \( L, M \) and \( KPQ \perp BD, OH \) meeting them in \( P, Q \) respectively (Fig. 112).

**Figure 112**

\[ \triangle LAK = \triangle DPK = \text{right angle and } \angle LAK = \angle BAC = \angle BDC = \angle KDP. \text{ Hence } \triangle LAK, \triangle PDK \text{ are similar. } \therefore AL/DP = AK/DK. \text{ But } \triangle KAC, KDB \text{ are also similar. } \therefore AK/AD = AC/BD = AF/DE \text{ (since } F, E \text{ are the mid-points of } AC, BD). \therefore AL/DP = AF/DE \text{ or } AL/AF = DP/DE. \therefore (AL + AF)/AF = (DP + DE)/DE \text{ or } KQ/AF = KM/DE. \therefore KQ/KM = AF/DE = FC/BE. \therefore \triangle s \]
BEG, CFH are similar, \( \therefore FC/BE = CH/BG \). \( \therefore \) \( \triangle KHQ \), KGM are similar, \( \therefore KQ/KM = KH/KG = CH/BG \). \( \therefore GH \parallel BC \) [Th. 4.83(ii)].

Again, \( \triangle AFH', DEG' \) are similar. \( \therefore AH'/DG' = AF/DE = CF/BE = CH/BG = KH/KG \). \( \therefore KA \cdot KG = KD \cdot KH \) (\( \because \) ADHG is cyclic), \( \therefore KH/KG = KA/KD = AH'/DG' \). \( \therefore G'H' \parallel AD \).

4.13. Describe a right-angled triangle of given perimeter such that the rectangle contained by the hypotenuse and one side will be equal to the square on the other side.

**Construction:** Let \( AB \) be the given perimeter. On \( AB \) describe a semi-circle \( AEB \). Find \( E \) such that if \( ED \) be drawn \( \perp AB \), \( EB = AD \). This is similar to dividing \( AB \) in \( D \) such that \( AD^2 = AB \cdot BD \) (see Problem 2.24) (\( AJ = \frac{1}{2} AB \) and \( JB = JK \)). Bisect \( \angle EAB \), \( EBA \) by \( AF, BF \). Draw \( FG \parallel AE \), and \( FH \parallel BE \) meeting \( AB \) in \( G, H \) (Fig. 113).

Hence \( FGH \) is the required \( \triangle \). (Note that \( \triangle GFH, AEB \) are similar.)

**Proof:** \( \angle GFH \) is right. Also, \( \angle GAF = \angle FAE \) = alternate \( \angle AFG \). \( \therefore GF = GA \). Similarly, \( HF = HB \). \( \therefore GH + GF + HF = AB = \) given perimeter. \( \therefore AB \cdot BE = AE^2 \); \( \therefore AB/AE = AE/BE \) and \( AB/BE = GH/GF \). Also, \( AE/BE = GF/FH \). \( \therefore GH/GF = GF/FH \).

Hence \( GF^2 = GH \cdot FH \).

4.14. In a triangle \( ABC \), \( D \) is the middle point of \( BC \). \( G, H \) are the points in which the inscribed and escribed circles touch \( BC \). \( E \) is the foot of the perpendicular from \( A \) to \( BC \), \( F \) the point in which the bisector of the angle \( A \) meets \( BC \). Prove that \( EG \cdot FH = EH \cdot FG \).

**Construction:** Let \( M, N \) be the centers of the inscribed and
escribed \( \odot \)s. Join \( FN, NH, NB, BM, MG \). Draw \( DR \perp BC \) meeting \( FN \) in \( R \) (Fig. 114).

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure114.png}
\caption{Figure 114}
\end{figure}

**Proof:** \( \because \) \( BM \) bisects \( \angle ABC \) and \( BN \) bisects \( \angle CBI \), and since \( AMN \) is a straight line bisecting \( \angle A \), \( \therefore \) in \( \triangle ABF \), \( \frac{AM}{MF} = \frac{AN}{NF} = \frac{AB}{BF} \) (Th. 4.90). In \( \triangle AEF \), \( \because \) \( MG \parallel AE \) (both \( \perp BC \)), \( \therefore \) \( AM/MF = EG/GF \). But, since \( \triangle s AFE, HFN \) are similar, \( \therefore \) \( AF/FN = FE/FH \). \( \therefore \) \( (AF + FN)/FN = (FE + FH)/FH \) or \( AN/FN = EH/FH \). \( \therefore \) \( EH/FH = EG/GF \). \( \therefore \) \( EG \cdot FH = EH \cdot FG \).

4.15. \( ABC \) is a right-angled triangle at \( A \), and \( AD \) is perpendicular to \( BC \). \( M, N \) are the centers of the circles inscribed in the triangles \( ABD, ACD \). If \( ME, NF \) are drawn parallel to \( AD \) meeting \( AB, AC \) in \( E, F \) respectively, then \( AE = AF \).

**Construction:** Let the \( \odot \)s \( M, N \) touch \( BC, AD \) in \( G, H \) and \( P, Q \) respectively. Join \( MG, MP, PB, MB, NQ, AN, AH, NH \) (Fig. 115).

**Proof:** \( \angle MBG = \frac{1}{2} \angle B = \frac{1}{2} \angle CAD = \angle NAQ \). \( \therefore \) right \( \angle s MBG, NAQ \) are similar. \( \therefore \) \( MB/MG = AN/NQ \). \( \therefore \) \( MB/MP = AN/NH \). Since \( \angle BMG = \angle ANQ \), \( \therefore \) \( \angle BMP = \angle ANH \). \( \therefore \) \( \angle BMP, ANH \) are similar. \( \therefore \) \( \angle MBP = \angle NAH \). \( \therefore \) \( \angle PBD = \angle HAD \). Hence \( \triangle s PBD, HAD \) are similar. \( \therefore \) \( BD/DP = AD/HD \). \( \because \) \( BD \cdot DH = AD \cdot DP = AD \cdot DG \). \( \therefore \) \( BD \cdot DH/CD = AD/CD = AD \cdot DG/CD \). \( \therefore \) \( BD \cdot DH(\text{CD} \cdot DG = AD/CD = AB/AC \). \( \therefore \) \( EMG \) is a straight line \( \parallel AD \). \( \therefore \) \( BD/ DG = AB/AE \). Similarly, \( DH/CD = AF/AC \). Multiplying, \( \therefore \) \( BD \cdot DH/CD \cdot DG = AB \cdot AF/AE \cdot AC = AB/AC \). \( \therefore \) \( AE = AF \).
4.16. ABC is a triangle and PQR is the triangle formed by the points of contact of the inscribed circle with AB, BC, CA respectively. Show that the product of the three perpendiculars drawn from any point in its circumference on the sides of triangle ABC is equal to the product of the other three perpendiculars drawn from that point on the sides of triangle PQR. If M is the center of this circle and BT, RS are perpendiculars, to AC, PQ respectively, then $BT \cdot MR = MB \cdot RS$.

**Construction:** Let D be any point on the inscribed $\odot M$. Draw $DE, DF, DG \perp AB, BC, CA$ respectively and $DH, DK, DL \perp PQ, QR, RP$ respectively. Join $DP, DR, LE, LG, MP$. Draw also $MN, MO \perp RS, BT$, and let $BM$ cut $PQ$ in $J$ (Fig. 116).

**Proof:** (i) Quadrilaterals $DEPL, DGRL$ are cyclic. $\therefore \angle DLE = \angle DPE$ and $\angle DGL = \angle DRL$. $\therefore \angle DPE = \angle DRL$ or $DRP$, $\therefore \angle DLE = \angle DGL$. Similarly, $\angle DEL = \angle DLG$. $\therefore \triangle DLE, DGL$ are similar. $\therefore DL/DE = DG/DL$. $\therefore DL^2 = DE \cdot DG$. Similarly,
\[ DH^2 = DE \cdot DF \text{ and } DK^2 = DF \cdot DG. \] Multiplying and taking square roots, \[ DL \cdot DH \cdot DK = DE \cdot DF \cdot DG. \]

(ii) \( \triangle BMO, \ RMN \) are similar. \( \therefore \ BM/RM = BO/RN, \) since \( \angle MPB \) is right and \( MB \) is \( \perp PQ \) at \( J. \) \( \therefore \ MP^2 = MJ \cdot MB = MR^2. \) \[ \therefore \ \frac{MB}{MR} = \frac{MR}{MJ} = \frac{BO}{RN}. \] 
\[ \frac{MB}{MR} = \frac{(MR + BO)}{(MJ + RN)} = \frac{(TO + BO)}{(NS + RN)} = BT/RS. \] \( \therefore \ BT \cdot MR = MB \cdot RS. \)

4.17. \( D, E \) are two points on the sides \( AB, \ AC \) of a triangle \( ABC \) such that \( \angle AED = \angle B. \ BE, CD \) intersect in \( F \) and \( BC, AF \) are bisected in \( M, \ N. \) If \( MN \) is produced to meet \( AB, \ AC \) or produced in \( R, \ Q \) respectively, show that \( AF \) is a common tangent to the circles circumscribing triangles \( ARQ, \ FRQ. \)

**Construction:** Bisect \( AB, \ AC \) in \( G, \ H \) and join \( NG, \ NH. \) Through \( N \) draw \( LNP, \ SNT \parallel \) to \( AB, \ AC \) meeting \( AC, GM, AB, MH \) in \( L, \ P, \ S, \ T \) respectively (Fig. 117).

**Figure 117**

\[ GN \parallel BF \text{ and } HN \parallel CF, \therefore \ \angle AGN = \angle ABF \text{ and } \angle AHN = \angle ACF. \] Since quadrilateral \( DBCE \) is cyclic, \( \therefore \ \angle ABF = \angle ACF. \) \( \therefore \ \angle AGN = \angle AHN. \) \( \therefore \ \triangle GSN = \angle A = \angle HLN, \therefore \ \triangle GSN, \ HLN \) are similar. \( \therefore \ NS/NS = GS/HL = NP/PM. \) \( \therefore \ NS/AS = NP/PM, \) since \( \angle ASN = \angle AGM = \angle NPM. \) Hence \( \triangle ASN, MPN \) are similar. \( \therefore \ \angle PMN = \angle NAS = \angle NQA (\text{since } GM \parallel CAQ). \) \( \therefore \ AN \) is tangent to \( \odot ARQ [\text{Th. 3.72(ii)}]. \) \( \therefore \ AN^2 = NR \cdot NQ = NF^2. \) \( \therefore \ NF \) is tangent to \( \odot FRQ (\text{Th. 3.80}); \) i.e., \( AF \) is a common tangent to \( \odot \)s circumscribing \( \triangle ARQ, \ FRQ. \)

4.18. Points \( D, E, F \) are taken in the sides \( BC, CA, AB \) respectively of a triangle \( ABC \) so that \( BD, CE, AF \) may be equal. Through \( D, E, F \) lines are drawn parallel to \( CA, AB, BC \) so as to form a triangle \( GHK \) in which \( KH \) is parallel to \( BC \) (Fig. 118). Show that \( (i) 2a - KH : 2b - GH : 2c - GK \)
\[ = a : b : c; \text{ (ii) area } GHK = \triangle ABC\left(2 - \left(\frac{p}{a} + \frac{p}{b} + \frac{p}{c}\right)\right)^2, \text{ where } a, b, c \text{ denote the sides } BC, CA, AB \text{ respectively and } p \text{ stands for } BD. \]

**Proof:** (i) \( \therefore \) \( \triangle s ABC, GKH \) are similar, \( \therefore \) \( BC/KH = AC/GH = AB/GK. \therefore \) \( (2 BC - KH)/BC = (2 AC - GH)/AC = (2 AB - GK)/AB. \) Hence \( 2a - KH : 2b - GH : 2c - GK = a : b : c. \]

(ii) \[ \triangle ABC\left(2 - \left(\frac{p}{a} + \frac{p}{b} + \frac{p}{c}\right)\right)^2 \]

\[ = \triangle ABC\left(2 - \left(\frac{BD}{BC} + \frac{CE}{CA} + \frac{AF}{AB}\right)\right)^2 \]

\[ = \triangle ABC\left(2 - \left(\frac{BD}{BC} + \frac{CL}{CB} + \frac{FM}{BC}\right)\right)^2 \]

\[ = \triangle ABC\left(2 - \left(\frac{BD}{BC} + \frac{CD + DL}{BC} + \frac{MH + FK + KH}{BC}\right)\right)^2 \]

\[ = \triangle ABC\left(2 - \left(\frac{BD + CD + DL + CD + BL}{BC} + \frac{KH}{BC}\right)\right)^2 \]

\[ = \triangle ABC \frac{KH^2}{BC^2} = \triangle ABC \frac{\triangle GHK}{\triangle ABC} = \triangle GHK. \]

**Figure 118**

4.19. Given two straight lines \( OA, OB \) and two points \( A, B \) in them, and a point \( P \) between them. It is required to draw through \( P \) a straight line, \( XPY \), so that \( XB \) is parallel to \( AY \).
CONSTRUCTION: Join \( OP \). Draw \( OD \) making an \( \angle DOA = \angle POB \). Take \( OD \) a fourth proportional to \( OA, OB, OP \); i.e., \( OD/OA = OB/OP \). Then draw on \( PD \) a \( \circ \) subtending an \( \angle = \angle DOB \). If this \( \circ \) does not cut \( OA \), it is impossible to solve the problem. If on the other hand, the \( \circ \) touches \( OA \), it has one solution, and if \( \circ \) cuts \( OA \) in two points, it has two solutions. Suppose \( \circ \) cuts \( OA \) in \( X, X' \); join \( XP \) and produce it to meet \( OB \) in \( Y \); then \( XB \) is \( || \) \( AY \) and similarly \( X'B \) is \( || \) \( AY' \) (Fig. 119).

\[ \text{FIGURE 119} \]

Proof: \( \therefore \angle OPY = \angle POX + \angle PXO = \angle DOB + PXO = \angle PXD + \angle PXO = \angle DXO \). Hence \( \triangle OPY, OXD \) are similar. \( \therefore OP/OY = OX/OD \). But \( OB/OP = OD/OA \) (construct). \( \therefore \) by multiplying, \( OB/OY = OX/OA \). \( \therefore XB \) is \( || \) \( AY \) [Th. 4.83(ii)]. Also, \( X'B \) is \( || \) \( AY' \).

4.20. If two homothetic triangles (similarly placed) are described such that one lies completely inside the other and if a third triangle can be constructed to circumscribe the smaller triangle and has its vertices on the sides of the larger triangle, then this third triangle will be the mean proportional of the first two.

CONSTRUCTION: Let \( ABC, DEF \) be the two homothetic \( \triangle s \) and \( GHK \) the third \( \triangle \). Join \( AE, AF \) and produce them to meet \( BC \) in \( M, N \); then join \( AD, FM \). Draw \( DL \parallel FE \) and join \( FL \) (Fig. 120).
Proof: \( \triangle DHF = \triangle DAF \) (Th. 2.39). Similarly, \( \triangle DKE = \triangle DAE \) and \( \triangle FGE = \triangle FME \). Hence \( \triangle GHK = \triangle AFM \). Also, \( \triangle DEF = \triangle LEF \).

\[
\frac{\triangle GHK}{\triangle DEF} = \frac{\triangle AFM}{\triangle DEF} = \frac{\triangle AFM}{\triangle LEF} = \frac{AM}{LE}
\]

\[
= \frac{\text{Alt. } \triangle AFM \text{ from } A \text{ on } BC}{\text{Alt. } \triangle LEF \text{ from } L \text{ on } FE} = \frac{BC}{FE}
\]

Since \( \triangle ABC \parallel \triangle GHK \), also \( \triangle AFM \parallel \triangle AMN \) \( = \frac{AF}{AN} = \frac{FE}{MN} \) and \( \triangle AMN \parallel \triangle ABC = MN/BC \). \( \therefore \triangle AFM \parallel \triangle ABC = FE/BC = \triangle GHK \parallel \triangle ABC = \triangle DEF \parallel \triangle GHK \). Hence the result is obvious.

4.21. A, B are the centers of two circles which touch externally. If the two common tangents CD, EF are drawn to the circles so that C, E lie on circle A, find the area of the trapezoid CEFD in terms of the radii.

Construction: From G the contact point of \( \odot s \), draw the third common tangent PGQ. Join AB, BF and draw AK, PR \perp s BF, FE respectively (Fig. 121).

Proof: Let \( r, r' \) be the radii of \( \odot s \) A, B respectively. Since PGQ bisects CD, EF, \( \therefore \triangle QCD + \triangle PEF = \text{trapezoid } CEFD = 2 \triangle PEF \).

\( \therefore \text{trapezoid } CEFD = PR \cdot EF \). \( \therefore AB = r + r' \) and \( BK = r' - r \), \( \therefore AK^2 = AB^2 - BK^2 = 4rr' \). Hence \( AK = 2 \sqrt{rr'} = EF \). Since \( \triangle s PQR, ABK \) are similar (GBFQ is cyclic), \( \therefore PR^2/PQ^2 = AK^2/AB^2 \).

\( \therefore EF = AK = PQ, \therefore PR^2/EF^2 = EF^2/AB^2 \). Therefore, \( PR^2/4rr' \)
= \frac{4rr'}{(r + r')^2} \text{ or } PR = \frac{4rr'}{(r + r')} \text{ Hence trapezoid } CEFD
= PR \cdot EF = 8 \frac{rr'}{\sqrt{r'}}(r + r').

4.22. From a point } P \text{ in the base } BC \text{ of a triangle } ABC, \ PM, \ PN \text{ are drawn perpendiculars to } AB, \ AC \text{ respectively and the parallelogram } AMNR \text{ is completed. Show that } R \text{ lies on a fixed straight line. }

\text{CONSTRUCTION: } \text{Draw } BE, CD \perp \text{s } AC, \ AB \text{ respectively. Complete the Os } ABES, \ ADCQ. \text{ Join } QR, QS \text{ and draw } CFG \parallel QR, \text{ meeting } NR, ES \text{ in } F, G. \text{ Then } QS \text{ is the fixed line on which } R \text{ lies (Fig. 122).}

\text{Proof: } \text{If it is shown that } QR \text{ coincides with } QS, \text{ then } QRS \text{ is one line. Now, } RF = QC = AD. \therefore FN = DM \text{ (since } RN = AM). \therefore GE/FN = EC/NC = BC/CP = BD/DM, \therefore GE = BD. \text{ Hence } SG = AD = QC. \therefore SGCG \text{ is a } \angle \therefore CG || QS. \text{ Since it is also } || QR \text{ (by construction), } \therefore QR \text{ coincides with } QS. \text{ But, } BE, CD \text{ are fixed alternates of } \triangle ABC. \text{ Hence } QRS \text{ is also fixed.
4.23. Circles are described on the sides of a quadrilateral $ABCD$, whose diagonals are equal, as diameters. Prove that the four common chords of pairs of circles on adjacent sides form a rhombus.

**Construction:** Let $ABCD$ be the original quadrangle and $EFGH$ be the quadrangle formed by the intersection of the common chords $BPEF, ASGF, ECQH, GDRH$. Join $PQ, QR, RS$ (Fig. 123).

**Proof:** Since $\odot$s on $AB, BC$ as diameters intersect on $AC$ ($\therefore \angle APB = \angle CPB = \text{right}$), then $P$ lies on $AC$ or produced. Similarly, for $\odot$s taken in pairs, their points of intersection lie on $AC, BD$.

$\therefore$ quadrangle $ARDS$ is cyclic, $\therefore \angle QSR = \angle CAD$. Similarly, quadrangle $QCDR$ is cyclic. $\therefore \angle SQR = \angle ACD$. Hence $\triangle QRS, CDA$ are similar. $\therefore CD/AC = QR/QS$. Also, $\triangle SQR, BCD$ are similar. $\therefore BD/CD = PR/QR$.

Multiplying yields $BD/AC = PR/QS$. $\therefore BD = AC, \therefore PR = QS$. Since also $\triangle HCR, HDQ$ are similar, $\therefore HD/QD = HC/CR$. But $\angle HQD = \angle QSA = \text{right}$. $\therefore HQ \parallel SG$. Similarly, $HR \parallel EP$. $\therefore HD/QD = HG/QS$ and $HC/CR = HE/PR$. $\therefore HG/QS = HE/PR$. $\therefore HG = HE. \therefore HE \parallel GF (\perp SB)$, $\therefore EFGH$ is a rhombus. Hence a $\odot$ can be inscribed inside the rhombus $EFGH$.

4.24. Given the vertical angle of a triangle, the ratio of the radii of the circles inscribed in the two triangles formed by the line bisecting the vertical angle and
meeting the base, and the distance between the centers of these circles, construct the triangle.

**Construction:** Let \( \angle BAC \) be the given vertical angle of \( \triangle \). Bisect \( \angle BAC \) by \( AP \) and \( \angle BAP, CAP \) by \( AQ, AR \). On \( AQ, AR \) take the distances \( AD, AE \) with the same given ratio of the radii. Between \( AD, AE \), take the distance \( MN = \) given distance between the centers of inscribed \( \odot Os \) and \( \parallel DE \). This is done by taking \( EL = \) this distance on \( DE \) and drawing \( LM \parallel AR \). On \( MN \) as diameter, construct a semicircle to cut \( AP \) in \( F \). Then draw \( FB, FC \) making \( \angle MFB = \angle MFA \) and \( \angle NFC = \angle NFA \) (Fig. 124). Therefore \( ABC \) is the required \( \triangle \).

![Figure 124](image)

**Proof:** Draw the two \( \odot Os \) with centers \( M, N \) and radii \( MG, NH \) (which are \( \perp AP \)). Now, \( \triangle MGJ, NHJ \) are similar, and \( AJ \) bisects \( \angle MAN \). \( \therefore \) \( MJ/JN = MG/NH = AM/AN = AD/AE = \) given ratio. Since \( FM, FN \) bisect \( \angle AFB, AFC, \therefore M, N \) are the centers of inscribed \( \odot Os \) in \( \triangle AFB, AFC \) and ratio of their radii \( MG : NH = \) given ratio. Again, \( \angle MFN = \) right. \( \therefore \) \( \angle AFB + \angle AFC = 2 \) right angles. Hence \( BFC \) is a straight line touching \( \odot Os M, N \) with \( MN = \) given distance.

4.25. \( ABCD \) is a quadrilateral inscribed in a circle. \( BA, CD \) produced and \( AD, BC \) produced meet in \( E, F \). If \( G, H, K \) are the middle points of \( BD, AC, EF \) and \( M \) is the intersection of \( BD, AC \), show that (i) \( AC/BD = 2 \) \( HK/EF = EF/2 \) \( GK \); (ii) \( GH/EF = (BD^2 - AC^2)/2 BD \cdot AC \). If also from \( M \) a line parallel to \( BC \) is drawn meeting \( AB, CD, AF, EF \) in \( P, Q, R, S \), prove that \( RS^2 = RP \cdot RQ \).
CONSTRUCTION. Join \( FH \) and produce it to \( N \) such that \( FH = HN \); then join \( AN, EN \) (Fig. 125).

**Figure 125**

Proof: (i) From congruence of \( \triangle AHN, CHF : AN = \triangle \) and || \( CF \). Hence \( \angle NAE = \) supp. of \( \angle NAB = \) supp. of \( \angle ABC = \angle ADC \) = \( \angle EDF \). In similar \( \triangle EAC, EDB : AC/BD = AE/ED \). Also in similar \( \triangle FDB, FCA : AC/BD = CF/DF \). \( \therefore \) \( AE/ED = CF/DF \). Hence \( ED/DF = AE/CF = AE/AN \). Therefore, \( \triangle ANE, DFE \) are similar (Th. 4.86). \( \therefore \) \( AC/BD = EN/EF \). Similarly, \( AC/BD = EF/2 GK \). Therefore, \( AC/BD = 2 HK/EF = EF/2 GK \).

(ii) Since the mid-points of the three diagonals of a complete quadrangle are collinear (Problem 2.16) and \( AC/BD = 2 HK/EF = EF/2 GK \),

\[
\frac{1}{BD} \left(\frac{AC}{BD} - \frac{AC}{BD} \right) = \frac{1}{2} \left(\frac{2 GK}{EF} - \frac{2 HK}{EF} \right) \text{ or } \frac{BD^2 - AC^2}{2 BD \cdot AC} = GH.
\]

(iii) \( SQ/PQ = CF/BC \). Also, \( QR/QM = CF/BC \). \( \therefore \) \( CF/BC = (SQ - QR)/(PQ - QM) = RS/PM \). Since \( CF/BC = RM/PM \) (in \( \triangle ABF \)), \( \therefore \) \( RM = RS \). \( \therefore \) \( RQ/RM = RM/RP = FC/FB \), \( \therefore \) \( RM^2 = RP \cdot RQ = RS^2 \).

4.26. If \( D, D' \) and \( E, E' \) and \( F, F' \) are the points where the bisectors of the interior and exterior angles of a triangle \( ABC \) meet the opposite sides \( BC, CA, AB \) or produced in pairs, show that (i) \( 1/DD' = 1/EE' + 1/FF' \);
(ii) \( a^2/\overline{DD'} = b^2/\overline{EE'} + c^2/\overline{FF'} \), where \( a, b, c \) are the lengths of the sides \( BC, BA \), respectively (Fig. 126).

**Figure 126**

Proof: (i) \( BD : C = CD : b \) [Th. 4.90(i)]. \( \therefore DB + CD : (b + c) = CD : b. \therefore CD = ab/(b + c). \) Similarly, \( CD' = ab/(c - b) \). Hence \( CD + CD' = DD' = 2abc/(c^2 - b^2) \). Likewise, it can be shown that \( EE' = 2abc/(c^2 - a^2) \) and \( FF' = 2abc/(a^2 - b^2) \). Assuming \( c > a \) and \( a > b \) and considering counterclockwise rotation to be \((+)\), then \( CB, BA, AC \) or \( D'D, FF', EE' \) have a \((+)\) sign. Therefore, \( DD' \) is a \((-)\) quantity. Hence

\[
\frac{1}{DD'} + \frac{1}{EE'} + \frac{1}{FF'} = \frac{1}{2abc} \left\{ b^2 - c^2 + c^2 - a^2 + a^2 - b^2 \right\} = 0.
\]

\( \therefore 1/DD' = 1/EE' + 1/FF' \).

(ii) Similarly,

\[
- \frac{a^2}{DD'} + \frac{b^2}{EE'} + \frac{c^2}{FF'} = \frac{1}{2abc} \left\{ a^2b^2 - a^2c^2 + b^2c^2 - b^2a^2 + c^2a^2 - c^2b^2 \right\} = 0.
\]

\( \therefore a^2/DD' = b^2/EE' + c^2/FF' \).

4.27. \( G, H \) are the middle points of the diagonals \( BD, AC \) of a quadrilateral \( ABCD \). If \( GH \) produced from both sides meets the sides \( AB, BC, CD, DA \) in \( E, F, J, K \) respectively, show that (i) \( AE : EB = FC : FB = CJ : JD = AK : KD \); (ii) \( HE : HF = HJ : HK = GE : GK = GJ : GF \) and if a circle with center \( M \) can be inscribed in the quadrilateral \( ABCD \), then
(iii) $ME : MJ = AB : CD$; (iv) $MF : MK = BC : AD$.

**Construction:** Draw the $\perp s$ $AN$, $BO$, $CL$, $DT$ on transversal $FEGHJK$ (Fig. 127).

![Diagram](image)

**Figure 127**

**Proof:** (i) Since $G$ is the mid-point of $BD$, then $\triangle s BOG, DGT$ are congruent. \( \therefore BO = DT \). Similarly, $AN = CL$. Now, $\triangle s BOE, ANE$ are similar. \( \therefore \frac{AE}{BE} = \frac{AN}{BO} \). Similarly, $\frac{FC}{FB} = \frac{CL}{BO}$ $\frac{CJ}{JD} = \frac{CL}{DT} = \frac{AN}{BO}$, and $\frac{AK}{KD} = \frac{AN}{DT} = \frac{AN}{BO}$. Hence $\frac{AE}{BE} = \frac{FC}{FB} = \frac{CJ}{JD} = \frac{AK}{KD}$.

(ii) Suppose that the previous ratios are denoted $(a)$. Join $AG, CG, BH, HD$. Now,

\[
\frac{HE}{HF} = \frac{\triangle BEH}{\triangle BFH} \frac{\triangle BCH}{\triangle BFH} = \frac{BE}{BC} = \frac{a - 1}{a + 1} \quad (\because \triangle ABH)
\]

Similarly, $\frac{HJ/HK = GE/GK = GJ/GF = (a - 1)/(a + 1)$. Hence these ratios are equal.

(iii) Let the incircle touch the sides $AB, BC, CD, DA$ in $P, Q, R, S$. Join $MA, MB, MC, MD, MP, MQ, MR, MS$. \( \because \triangle BEM/\triangle DJM = ME\cdot BO/MJ\cdot DT = ME/MJ \). Similarly, $\triangle AEM/\triangle CJM = ME/MJ$. Hence

\[
\frac{\triangle BEM + \triangle AEM}{\triangle DJM + \triangle CJM} = \frac{\triangle AMB}{\triangle DMC} = \frac{ME}{MJ} = \frac{MP\cdot AB}{MR\cdot CD} = \frac{AB}{CD}
\]

(since $MP = MR$).
(iv) In a similar way to (iii),

\[
\frac{\triangle FMC - \triangle BFM}{\triangle MAK - \triangle MDK} = \frac{\triangle BMC}{\triangle AMD} = \frac{MF}{MK} = \frac{MQ \cdot BC}{MS \cdot AD} = \frac{BC}{AD}
\]

(since \(MQ = MS\)).

4.28. If \(O\) is the center of the circumscribed circle of the triangle \(ABC\), \(I\) is the center of the inscribed circle, \(I_a\) is the center of the escribed circle opposite to \(A\), \(R, r, r_a\) are the radii of these three circles in order, and \(N\) is the center of the nine-point circle, then

(i) \(OI^2 = R^2 - 2Rr\);
(ii) \(OI_a^2 = R^2 + 2Rr_a\);
(iii) \(IN = \frac{1}{2} R - r\);
(iv) \(I_aN = \frac{1}{2} R + r_a\);
(v) \(R^2 - OG^2 = \frac{1}{9} \{AB^2 + BC^2 + CA^2\}\), where \(G\) is the centroid of the triangle \(ABC\).

**Construction:** (i) Join \(III_a\) cutting the \(OO\) in \(D\). \(DO\) produced meets the circumference of \(OO\) in \(E\). Join \(CE, CD, IF\) (\(F\) point of contact of \(OI\) and \(AC\)) and produce \(OI\) from both sides to meet the circumference in \(M, X\) (Fig. 128).

![Figure 128](image)

**Proof:** \(\triangle s AIF, EDC\) are similar. \(\therefore AI/DE = IF/CD. \therefore AI \cdot CD = IF \cdot DE. \therefore CD = DI\) (see Problem 3.10), \(\therefore AI \cdot DI = 2Rr\). Since \(AI \cdot DI = MI \cdot IX = OM^2 - OI^2\), \(\therefore OI^2 = R^2 - 2Rr\).

(ii) Produce \(I_aO\) to meet the circumference of \(OO\) in \(J, L\) and join \(I_aK\) (\(K\) point of contact of \(OI_a\) and \(AC\) produced). Again, \(\triangle s AII_aK\),
EDC are similar. \[ AI_aDE = KI_aCD. \] \[ AI_aCD = KI_aDE = 2Rr_a. \] \[ CD = DI_a. \] \[ AI_a \cdot DI_a = I_aJ \cdot J_aL = OI_a^2 - R^2. \] \[ OI_a^2 = R^2 + 2Rr_a. \]

(iii) \( H \) is the orthocenter of \( \triangle ABC \). Join \( MH, OH \) and produce the latter to meet \( \odot O \) in \( P, Q \). Produce alt. \( AT \) to meet \( \odot O \) also in \( S \).

\[ N \] is the middle point of \( OH \) [Th. 3.77(i)], \[ \because \] in \( \triangle OIH \), \[ OI^2 + HI^2 = 2IN^2 + 2NH^2. \] \[ \therefore OI^2 = R^2 - 2Rr \] and \[ HI^2 = 2r^2 - AH \cdot HT, \] \[ 2HI^2 = 4r^2 - AH \cdot HS \] and \[ 2NH^2 = \frac{1}{2} OH^2 = \frac{1}{2} R^2 \] \[ - AH \cdot HT. \] \[ \therefore \] By adding, \[ 2(R^2 - 2Rr) + 4r^2 - AH \cdot HS = 4IN^2 + R^2 - AH \cdot HS, \] from which \[ IN = \frac{1}{2} R - r. \]

(iv) Join \( H I_a \). In \( \triangle OI_aH \), \[ OI_a^2 + HI_a^2 = 2I_aN^2 + 2NH^2. \] \[ \therefore 2(R^2 + 2Rr_a) + 4r_a^2 - AH \cdot HS = 4I_aN^2 + R^2 - AH \cdot HS, \] from which \[ I_aN = \frac{1}{2} R + r_a. \]

(v) \( G \) is the point of trisection of \( OH \) [Th. 3.77(ii)]. Since \[ AO^2 + BO^2 + CO^2 = AG^2 + BG^2 + CG^2 + 3OG^2 \] (easily proved), \[ \because 3R^2 - 3OG^2 = AG^2 + BG^2 + CG^2 = \frac{1}{3}(AB^2 + BC^2 + CA^2). \] Hence \[ R^2 - OG^2 = \frac{1}{9}(AB^2 + BC^2 + CA^2). \]

4.29. \( AA', BB', CC' \) are the altitudes of a triangle \( ABC \). \( D, E, F \) are the centers of the inscribed circles in the triangles \( AB'C', BC'A', CA'B' \). If the inscribed circle in the triangle \( ABC \) touches \( BC, CA, AB \) in \( L, M, N \) respectively, show that the sides of the triangle \( DEF \) are equal and parallel to those of triangle \( LMN \).

Construction: Produce \( LO \) to meet \( AB \) in \( G \) (\( O \) being the incenter of \( \triangle ABC \)). Join \( ADO, BEO, CFO, NE, MF, EA' \) (\( ADO, \) etc., are straight lines, since they bisect \( \angle s \) \( A \), etc.) (Fig. 129).

**Proof:** Quadrangle \( AC'A'C \) is cyclic. \[ \therefore \] \( \angle C'A'B = \angle A. \) Since \( EA' \) bisects \( \angle C'A'B \) and \( ADO \) bisects \( \angle A, \angle EA'B = \angle BAO = \frac{1}{2} \angle A. \)
Since, also, $BEO$ bisects $\angle B$, $\therefore \triangle ABO, \triangle A'BE$ are similar. $\therefore BA'/BA = BE/BO$. $\therefore LOG \parallel AA'$ (both being $\perp BC$), $\therefore BL/BG = BA'/BA$. $\therefore BL = BN$, $\therefore BN/BG = BA'/BA = BE/BO$. $\therefore NE \parallel LOG$. In $\triangle BOG$, $\therefore BN/BG = NE/GO = BL/BG = LO/GO$ ($BEO$ bisects $\angle B$). $\therefore NE = LO$. Hence $NE = \text{and} \parallel LO$. Similarly, $MF = \text{and} \parallel LO$ or $NE$. Therefore, $NEFM$ is a $\square$. $\therefore FE = \text{and} \parallel MN$. Similarly with $DF$, $NL$ and $DE$, $ML$.

4.30. In a triangle $ABC$, $Q$ is the point of intersection of $AF$ and $BN$ where $F$ and $N$ are the points of contact of the escribed circles opposite $A$ and $B$ with $BC$ and $AC$ respectively, $I$ the center of the inscribed circle, $D$ the middle point of $BC$, $G$ the centroid, $H$ the orthocenter, $O$ the center of the circumscribed circle. If $DI$ is produced to meet $AH$ in $E$, show that (i) $AE = \text{radius of inscribed circle}$; (ii) $AQ$ is parallel and equal to $2ID$; (iii) $Q, G, I$ are collinear and, $QG = 2IG$; (iv) $H, Q, O, I$ are the vertices of a trapezoid one of whose parallel sides is double the other and whose diagonals intersect in $G$.

**Construction:** (i) Let $\odot's I, M$ touch $AC, BC$ or produced in $K, L$ and $J, F$. Join $IK, ML, IJ, MF, AIM$. Produce $JI$ to meet $AF$ in $X$ (Fig. 130).

**Proof:** In $\triangle AML$, $\therefore IK \parallel ML$, $\therefore AI/AM = IK/ML$, since $MF \parallel JX (\perp BC)$. $\therefore$ In $\triangle AMF$, $AI/AM = IX/MF$. $\therefore IX/MF$
\[ \frac{IK}{ML} \implies MF = ML, \implies IX = IK. \text{ Hence } X \text{ lies on the inscribed } \odot I. \implies BF = CJ \text{ and } D \text{ is the mid-point of } BC, \implies D \text{ is also the mid-point of } FJ. \]

Also \( I \) is the mid-point of \( JX \). \( \therefore DI \parallel AF. \) But \( JX \parallel AHP. \implies AXIE \) is a \( \square \). \( \therefore AE = IX = \text{radius of } \odot I. \)

(ii) Bisect \( AC \) in \( R \) and join \( IR, RD. \implies DI \parallel AF \) and similarly \( RI \parallel BN \) and since \( RD \parallel AB, \implies \triangle IRD, QBA \) are similar. Therefore, \( ID/AQ = RD/AB = \frac{1}{2} \) or \( AQ \) is \( \parallel \) and \( = 2\, ID. \)

(iii) Join \( AD \) cutting \( QI \) in \( G. \) \( \triangle AGQ, DIG \) are similar. \( \therefore AQ/ DI = AG/DG = 2 \) [from (ii)]. Hence \( AG = 2\, DG, \) but \( AD \) is a median in \( \triangle ABC. \implies G \) is the centroid and \( Q, G, I \) are collinear. Since also \( AQ/ DI = QG/IG = 2, \implies QG = 2\, IG. \)

(iv) According to \( [\text{Th. 3.77(ii)}], O, G, H \) are collinear and \( HG = 2\, GO. \) Since \( QG = 2\, IG, \implies \triangle GQH, GIO \) are similar \( (\text{Th. 4.86}). \implies HQ = 2\, IO \) and \( \parallel \) to it, since the diagonals \( QI, HO \) intersect in \( G. \) Hence \( HQOI \) is the required trapezoid.

**Miscellaneous Exercises**

1. On the sides \( AB, AC \) of a triangle \( ABC, \) points \( D, E \) are taken such that \( AD \) is to \( DB \) as \( CE \) to \( EA. \) If the lines \( CD, BE \) intersect in \( F, \) the triangle \( BFC \) will be equal in area to the quadrilateral \( ADFE. \)

2. If \( O \) is the center of the inscribed circle of the triangle \( ABC, \) and \( AO \) meets \( BC \) in \( D, \) prove that \( AO : OD = (AB + AC) : BC. \)

3. From any point on the circumference of a circle perpendiculars are drawn to two tangents to the circle and their chord of contact. Prove that the perpendicular on the chord is a mean proportional between the other two perpendiculars.

4. The inscribed circle touches the side \( BC \) of the triangle \( ABC \) in \( D. \) An escribed circle touches \( BC \) in \( E. \) Show that the foot of the perpendicular from \( E \) on \( AD \) lies on the escribed circle.

5. A straight line is drawn from a vertex of a triangle to the point where the escribed circle touches the base. Show that if tangents be drawn to the inscribed circle at the points of intersection, one of them will be parallel to the base.

6. \( ABEC \) is a straight line harmonically divided; i.e., \( AB : AC = BE : EC. \) If through \( E \) a straight line is drawn meeting parallels through \( B, C \) in \( D \) and \( Q, \) and if \( AQ \) cuts \( DB \) produced in \( P, \) show that \( BP = BD. \)

7. If a circle \( PDG \) touches another circle \( ABC \) externally in \( D \) and a chord \( AB \) extended in \( P, \) and if \( CE \) is perpendicular to \( AB \) at its middle point \( E \) and on the same side of \( PAB \) as the circle \( PDG, \) prove that the rectangle contained by \( CE \) and the diameter of \( PDG = PA \cdot PB. \)
8. \(ABC, DEF\) are two straight lines \(AD, BE, CF\) being parallel to one another. Prove that if \(AF\) passes through the middle point of \(BE\), then \(CD\) will also pass through that point.

9. \(ABCD\) is a rectangle inscribed in a circle. \(AD\) is produced to any point \(E. EC\) is joined and produced to cut the circle and \(AB\) produced in \(F, G\) respectively. Prove that \(AG \cdot AE : FB \cdot FD = EG^2 : BD^2\).

10. If \(AD, BE, CF\) are any three concurrent straight lines drawn from the vertices of a triangle \(ABC\) to the opposite sides and \(M\) is their point of concurrency, then \(DM : DA + EM : EB = FM : FC = 1\).

11. \(DE\) is any straight line parallel to the side \(AC\) of a triangle \(ABC\) meeting \(BC, AB\) in \(D, E\). If a point \(F\) is taken on \(AC\) such that the triangles \(ABF, BED\) are equal and \(FG\) is drawn parallel to \(AB\) meeting \(BC\) in \(G\), show that \(BD\) is a mean proportional between \(BC, BG\). (Join \(AG, GE, AD\) and show that \(GE \parallel AD\).)

12. \(ABCD\) is a quadrilateral inscribed in a circle and \(E\) is the intersection of the diagonals. Show that \(BE : DE = AB \cdot BC : AD \cdot CD\).

13. If a circle is drawn passing through the corner \(A\) of a parallelogram \(ABCD\) and cutting \(AB, AC, AD\) in \(P, Q, R\) respectively, then \(AQ \cdot AC = AP \cdot AB + AR \cdot AD\). (Join \(PR, QR\) and draw \(CG\) making with \(AD\) produced \(\angle DGC = \angle AQR\). Another solution apply Ptolemy’s theorem to \(APQR\).)

14. Circles are drawn on the sides of a right-angled triangle \(ABC\) at \(A\), as diameters. From \(A\) any transversal is drawn cutting circles on \(AB, BC, CA\) in \(F, G, H\). Show that \(FG = AH\).

15. \(ABC\) is a triangle and lines are drawn through \(B\) and \(C\) to meet the opposite sides in \(E, F\). If \(BE, CF\) intersect in a point on the median from \(A\), show that \(EF\) is parallel to \(BC\).

16. Show that if on the sides of a right-angled triangle \(ABC\), similar triangles are described so that their angles opposite to the sides \(AB, BC, CA\) are equal, then the triangle on the hypotenuse is equal to the sum of the other two triangles. (Let \(\angle A = \) right angle, \(ABD, BCE, ACF\) be the equiangular \(\triangle s\). Draw \(AG \perp BC\) and join \(GE\). Then show that \(\triangle ABD = \triangle BGE\).)

17. In the triangle \(ABC, AD, BE, CF\) are the altitudes and \(EM, FN\) are the perpendiculars from \(E, F\) on \(BC\). Show that \(\triangle CED : \triangle BFD = CM : BN\).

18. Show that the rectangle contained by the perpendiculars from the extremities of the base in a triangle to the external bisector of the vertex angle is equal to the rectangle contained by the perpendiculars from the middle point of the base to the same external bisector and to the internal bisector of the vertex angle.

19. The side \(AKLB\) of a rectangle \(ABCD\) is three times the side \(AD\). If \(K, L\) are the points of trisection of \(AB\), and \(BD\) meets \(KC\) in \(R\), prove that \(C, R, L, B\) lie on a circle.
20. *ABCD* is a quadrilateral inscribed in a circle. If the bisectors of the angles *CAD*, *CBD* meet in *G*, show that \( \frac{AG}{BG} = \frac{AD + AC}{BD + BC} \).

21. *ABCD* is a trapezoid whose parallel sides *CD*, *AB* have a ratio of 2 : 5. If *DE*, *CL* are drawn parallel to *BC*, *AD* meeting *AB* in *E*, *L* and *DE*, cutting *AC*, *CL* in *H*, *G*, show that \( \triangle CGH = \frac{8}{105} \) trapezoid *ABCD*.

22. Show that the lines joining the vertices of a triangle to the points of contact of the inscribed circle with the opposite sides are concurrent and that these lines are bisected by the lines joining the middle points of the opposite sides to the center of the incircle.

23. *ABC* is a triangle inscribed in a circle. If the altitudes *AD*, *BE*, *CF* are produced to meet the circle in *X*, *Y*, *Z* respectively, show that \( \frac{AX}{AD} + \frac{BY}{BE} + \frac{CZ}{CF} = 4 \). (Join *BX* and use Exercise 10.)

24. If \( a, b, c \) are the lengths of the sides of a triangle *ABC* and if the internal bisectors of the angles *A*, *B*, *C* meet *BC*, *CA*, *AB* in *X*, *Y*, *Z* and assuming \( s \) is half the perimeter of the triangle, prove that (a) \( AX^2 = bc \left\{ 1 - \frac{a^2}{(b + c)^2} \right\} \); (b) \( AX \cdot BY \cdot CZ = 8abc \cdot \triangle ABC \cdot (a + b)(b + c)(c + a) \); (c) \( \triangle XYZ/\triangle ABC = \frac{2 abc}{(a + b)(b + c)(c + a)} \).

25. Find the radius of the circle inscribed in a rhombus whose diagonals are \( 2a \) and \( 2b \).

26. *M*, *N* are the centers of two circles intersecting in *A*, *B*. From any point *C* on the circumference of either one of the circles, a tangent *CD* is drawn to the other. Show that \( (CD^2 : CA \cdot CB) \) is constant. (Let \( c \) be on circle *N*, *CD* tangent to circle *M*. Produce *CA* to meet circle *M* in *E*, then join *EB*, *AB*, *MN*, *BM*, *BN").

27. Divide a given arc of a circle into two parts so that the chords of the parts are in a given ratio.

28. Through two fixed points on the circumference of a circle draw two parallel chords which will be to each other in a given ratio.

29. *ABC* is a triangle inscribed in a circle. From *A* straight lines *AD*, *AE* are drawn parallel to the tangents at *B*, *C* respectively, meeting *BC* produced if necessary in *D*, *E*. Prove that \( BD : CE = AB^2 : AC^2 \).

30. In Problem 2.8, show that *AM* is a mean proportional between *MC*, *BN*.

31. *ABC* is an equilateral triangle inscribed in a circle and *D* any point on the circumference. If *BD*, *AD* produced meet *AC*, *CB* or produced in *E*, *F* and *DC* cuts *AB* in *G*, prove that (a) triangles *ABF*, *CDF*, *BAE*, *CDE* are similar; (b) \( \frac{BG}{CE} = \frac{CG^2}{BE^2} \).

32. Prove by ratio and proportion that the middle points of the three diagonals of a complete quadrilateral are collinear. (See Problem 2.16. Complete \( \square s AECK, EBLD \). Join *EK*, *EL*, *DL*, *FL*, *LK*. Prove that *KLF* is a straight line. Hence *GHJ* is one line \( || \) *KLF*.)
33. **DM, DR are two tangents to a circle from a point D.** From D a line is drawn parallel to another tangent from any point A on the circle, meeting AM, AR produced in B, C. Show that BMRC is cyclic, and find the center of its circle.

34. In an equilateral triangle draw a straight line parallel to one of the sides so as to divide the triangle into two parts whose areas are proportional to the squares on lines equal to their perimeters.

35. **AB** is a diameter of a semi-circle whose center is O. **AO** is bisected in C and on AC, CB as diameters two semi-circles are inscribed in the first one. If **DE** is a common tangent to the smaller semi-circles and produced to meet **BA** produced in **M**, show that **AC = 2 AM**.

36. **CD, CE** are tangents to a circle from any point **C**. **AB** is a chord in the circle bisected by **DE**. If **AG, AH, BM, BN** are perpendiculars to both tangents **CD, CE** from **A, B**, show that **AG·AH = BM·BN**. (Use Exercise 3.)

37. **ABC** is a triangle. Two circles are described passing through **B, C** such that one touches **AB** in **B** and the other touches **AC** in **C**. If the circles cut **AB, AC** or produced in **F, G**, show that (a) **BG || CF**; (b) **AG : AF = AB^3 : AC^3**.

38. (a) Construct a triangle given the base, the area, and the sum of the sides; (b) construct a triangle given the base, the area, and the difference of the sides. (Use Problem 3.25.)

39. **ABCD** is a quadrilateral and **F, G, H** are three points on **AD, BD, CD** respectively such that **AF : FD = BG : GD = CH : HD**. If **M, N, L** are the middle points of **AB, AC, BC** respectively, prove that **FL, GN, HM** are concurrent. (Join the sides of the triangles **FGH, MNL** and show that they are homothetic.)

40. A straight line bisects the base **BC** of a triangle **ABC**, passes through the center of the inscribed circle, and meets at **P** a straight line drawn through **A** parallel to the base. Show that **AP = \frac{1}{2} (AB - AC)**.

41. A circle cuts the sides **BC, CA, AB** of a triangle **ABC** in six points **A', A'', B', B'', C', C''** respectively and the perpendiculars to the respective sides at three of these points are concurrent. Show that those erected at the other three points are also concurrent.

42. From the vertex **A** of a triangle **ABC**, **AD** and **AE** are drawn to the base making with **AB, AC** two equal angles. Show that **AB^2 : AC^2 = BD·BE : CD·CE**.

43. Two straight lines **BP, CQ** are drawn from **B, C** of a triangle **ABC** to meet the opposite sides in **P, Q** and intersect on the altitude **AD** in **E**. Show that **AD** bisects the angle **PDQ**.

44. From a given point on the circumference of a given circle, draw two chords so as to be in a given ratio and to contain a given angle.
45. A point $D$ is taken in the side $AB$ of a triangle $ABC$, and $DC$ is drawn. It is required to draw a straight line $EF$ parallel to $BC$ and meeting $AB$, $AC$ in $E$, $F$ so that the quadrilateral $EBCF$ may be equal to the triangle $DBC$. (In $AB$ take $AE$ the mean property between $AB$, $AD$ and draw $EF \parallel BC$.)

46. Divide a triangle into any number of equal parts by straight lines parallel to the base.

47. $ABCD$ is a parallelogram; $APQ$ is drawn cutting $BC$ and $DC$ produced in $P$, $Q$. If the angle $ABP'$ be made equal to the angle $ADQ'$, $BP' = BP$ and $DQ' = DQ$, prove that the angles $PBP'$, $QDQ'$, $P'AQ'$ are all equal and that $AP' : AQ' = AP : AQ$.

48. If two of the sides of a quadrilateral are parallel, show that the difference of the squares on the two diagonals is to the difference of the squares on the non-parallel sides as the sum of the lengths of the parallel sides is to their difference.

49. $ABCD$ is a rhombus and any straight line is drawn from $C$ to cut $AB$, $AD$ produced in $F$, $G$. Show that $\frac{1}{AB} = \frac{1}{AF} + \frac{1}{AG}$.

50. $D$, $E$ are the points of intersection of two circles of which one is fixed and the other passes always through fixed points $A$, $B$. Prove that the ratio $(AD \cdot AE : BD \cdot BE)$ is constant. (Produce $AD$, $BE$ to cut a fixed circle in $L$, $R$. Join $LE$, $RD$.)

51. $AD$, $BE$, $CF$, the perpendiculars from the vertices of a triangle on the opposite sides, intersect in $O$. Prove that $AO \cdot AB = AE \times$ diameter of the circumscribed circle and $OB \cdot OC = OD \times$ diameter of the circumscribed circle.

52. Construct a quadrilateral $ABCD$ given the four sides and the area. [Produce $BC$ to $E$ such that $BC/BE = AD/AB$. Draw $CG$, $EF \perp AB$, and $CH \perp AD$. Prove that $2AD(BF + DH) = AB^2 + BC^2 - AD^2 - DC^2 = \text{given}$. $\because (BF + DH)$ is given. Also, $2 \triangle ABCD = AD(CH + EF)$. $\therefore (CH + EF)$ is given, also $BE$ is given.]

53. $AB$, $AC$ are the sides of a regular pentagon and a regular decagon inscribed in a circle with center $O$. The angle $AOC$ is bisected by a straight line which meets $AB$ in $D$. Prove that the triangles $ABC$, $ACD$ are similar, also the triangles $AOB$, $DOB$. Thence prove that $AB^2 = AC^2 + AO^2$.

54. From the vertex of a triangle draw a line to the base so that it may be a mean proportional between the segments of the base. (Describe a circle about given $\triangle ABC$ and find its center $O$. On $OA$ describe a semicircle $ODA$ cutting $BC$ in $D$. Draw chord $ADE$ and show that $AD^2 = AD \cdot DE = BD \cdot DC$.)

55. Two circles touch externally in $C$. If any point $D$ is taken without them such that the radii $AC$, $BC$ subtend equal angles at $D$ and $DE$, $DF$ are tangents to the circles, then $DE \cdot DF = DC^2$. 
56. Two triangles which have one angle of the one equal to one angle of the other, and are to one another in the ratio of the squares of the sides opposite those angles, are similar to each other.

57. Two circles touch in $O$, and a straight line cuts one in $A, B$, the other in $C, D$. Show that $OB \cdot OC : OA \cdot OD = BC : AD$. (Draw the common tangent $OK$ meeting $AB$ in $K$ and $CE$, $AF \perp BO$, $DO$ respectively.)

58. From the vertex $A$ of a triangle $ABC$ inscribed in a circle, a tangent $AD$ is drawn to touch the circle at $A$ and meet the base $BC$ produced in $D$. From $D$ another tangent $DE$ is drawn touching the circle at $E$. $BML$ is drawn parallel to $AD$ meeting $AE$, $AC$ or produced in $M$, $L$. Show that $BM = ML$.

59. $E$ is the middle point of the side $BC$ of a square $ABCD$, $O$ is the intersection of its diagonals, $F$ the middle point of $AE$, $G$ the centroid of the triangle $ABE$. If $OG$ cut $AE$ in $H$, prove that the square $ABCD = 8 \cdot AE \cdot FH$.

60. $AB$, the diameter of a circle, is trisected in $C, D$. $PCQ$, $PDR$ are two chords of the circle and $QR$ meets $AB$ produced in $N$. Prove that $PC^2 : PD^2 = NR : NQ$. (From $Q$ draw $QF \parallel PDR$ meeting $AB$ in $F$.)

61. Construct a triangle having given the radius of its circumscribed circle and the radii of two of the four circles touching the sides.

62. $ABC$ is a triangle. Find a point $D$ on $AB$ such that if a parallel $DE$ to the base $BC$ be drawn to meet $AC$ in $E$, then $DE^2 = BD^2 + CE^2$.

63. Construct a triangle on a given base and with a given vertical angle such that the base may be a mean proportional between the sides. Show that the problem is possible only when the given angle is not greater than $60^\circ$.

64. Show that any two diagonals of a regular pentagon cut each other in extreme and mean ratio.

65. $ABC$ is a triangle inscribed in a circle. If the bisector of the vertex angle $BAC$ meets the base $BC$ in $D$ and the circumference in $E$ and is bisected in $D$, show that $AB^2 = 2BD^2$ and $AC^2 = 2CD^2$.

66. Construct a triangle given the base and the vertex angle so that the rectangle contained by the sides may be a maximum.

67. Construct a triangle given the base, the vertical angle, and the ratio between the perimeter and the altitude to that base.

68. A circle is described about an isosceles triangle. Prove that the distance of any point in the arc subtended by the base opposite the vertex from the vertex bears a constant ratio to the sum of its distances from the other two vertices.

69. Through a given point $P$ in the base $BC$ of a triangle $ABC$, draw a straight line to cut the sides $AB$, $AC$ in $R$, $Q$ respectively so that $BR$ will be equal to $CQ$. 

70. Three circles touch each other internally at the same point. Prove that the tangents drawn from any point on the largest circle to the other two circles bear to one another a constant ratio.

71. Prove that the side of a regular polygon of twelve sides inscribed in a circle is a mean proportional between the radius of the circle and the difference between the diameter of the circle and the side of an equilateral triangle inscribed in the circle. (Let $ABC$ be the equilateral triangle inscribed in circle $O$. Draw diameter $AOD$ and join $BD$. Bisect arc $BD$ in $F$ and join $FB, FO, BO$. $FB$ will be the side of a twelve-sided regular polygon.)

72. $ABC$ is a triangle; the sides $AB, AC$ are cut proportionally in the points $D, E$. From any point $P$ in $BC$ two straight lines $PQ$ and $PR$ are drawn meeting $AB$ or $AB$ produced in $Q, R$, and always intercepting a portion $QR$ which is equal to $AD$. Also, $PQ', PR'$ are drawn meeting $AC$ or $AC$ produced in $Q', R'$, and always intercepting a portion $Q'R'$ which is equal to $AE$. Show that the sum of the areas $PQR, PQ'R'$, is constant.

73. Prove that the straight line drawn from a vertex of a triangle to the center of the inscribed circle divides the line joining the orthocenter to the center of the circumscribed circle into segments, which are in the ratio of the perpendicular from the center of the circumscribed circle on the opposite side of the triangle to the radius of the nine-point circle.

74. Three circles have a common chord, and from any point in one, tangents are drawn to the other two. Prove that the ratio of these tangents is constant.

75. From the point of contact of two circles which touch internally are drawn any two chords at right angles, one in each circle. Prove that the straight line joining their other extremities passes through a fixed point. (Let $\odot ABE$ with center $O$ touch $\odot ADC$ with center $O'$ internally at $A$. Let $AE, AB$ be chords $\perp$ each other. Produce $AB$ to $D$; $BE, DC$ are diameters, and let $BC$ cut $OO'$ in $G$. Show that $G$ is a fixed point.

76. $AD$ is drawn perpendicular to the hypotenuse $BC$ of a right-angled triangle $ABC$. On $BC$, $AB$ similar triangles $BEC$, $AGB$ are similarly described so that the angles $CBE, ABG$ are equal; $DE$ is drawn. Prove that the triangles $ABG, BDE$ are equal.

77. From a point $T$ a tangent is drawn to each of two concentric circles, and through the common center $C, CR'$ is drawn parallel to the bisector of the exterior angle at $T$, meeting the tangents in $R, R'$. Show that the ratio of $CR$ to $CR'$ is independent of the position of $T$.

78. $ABD$ is the diameter of a semi-circle $ACD$, and $ABC$ is a right angle. $E$ any point on the chord $AC$ inside the semi-circle is joined to $B$, and $CF$ is drawn cutting $AD$ in $F$ and making the angle $BCF$ equal to the angle $ABE$. Prove that $AE : EC = BF : BD$. 


79. Show how to draw through a given point in a side of a triangle a straight line dividing the triangle in a given ratio. (In $BA$ take $BD$ so that $BD : AB$ in the given ratio. Let $D$ be between $B$ and $P$ the given point in $AB$. Draw $DE \parallel CP$; $DE$ will be the required line.)

80. Two parallelograms $ABCD, A'B'C'D'$ have a common angle at $B$. If $AC'$ and $DD'$ meet in $O$, prove that $OD' : OD = \text{fig. } A'B'C'D' : \text{fig. } ABCD$.

81. $A, B$ are the centers of two circles and $DEFG$ is a transversal cutting the circles $A, B$ in $D, E$ and $F, G$ respectively such that the ratio $DE : FG$ is fixed. Two tangents $DP, GP$ are drawn to the circles $A, B$ respectively. Show that $DP : GP$ is a constant ratio.

82. Given the three altitudes of a triangle, construct the triangle.

83. Construct a triangle equal to a given triangle and having one of its angles equal to an angle of the triangle and the sides containing this angle in a given ratio.

84. Two circles intersect in $A, B$. The chords $BC, BD$ are drawn touching the circles at $B$, and the points $D, C$ are joined to $A$. Prove that $AC, AD$ are to each other in the ratio of the squares of the diameters of the circles.

85. Two fixed circles touch externally at $A$, and a third passes through $A$ and cuts the other two orthogonally in $P, Q$. Prove that the straight line $PQ$ passes through a fixed point.

86. If the base of a triangle be a mean proportional between the sides, prove that the bisectors of the angles at the base will cut off on the sides segments measured from the vertex such that their sum is equal to the base.

87. If perpendiculars be drawn from any point on the circumference of a circle to the sides of an inscribed quadrilateral, the rectangle contained by the perpendiculars on two opposite sides is equal to the rectangle contained by the other two perpendiculars.

88. Prove that any straight line drawn from the orthocenter of a triangle to the circumference of the circumscribing circle is bisected by the nine-point circle.

89. The opposite sides $AB, DC$ of a quadrilateral $ABCD$ are divided in a given ratio at $E$ and $F$ so that $AE : EB = DF : FC$, and the other pair of opposite sides $BC, AD$ are divided at $G, H$ in another given ratio so that $BG : GC = AH : HD$. Show that the point of intersection of $EF$ and $GH$ divides $GH$ in the first of the given ratios and $EF$ in the second.

90. Construct a triangle similar to a given triangle and having its vertices on three given parallel lines.
91. Divide a quadrilateral in a given ratio by a straight line drawn from a given point in one of its sides. (Let $ABCD$ be the given quadrilateral and $P$ given point in $CD$. Convert $ABCD$ into an equal triangle in area through $P$ by drawing $CE \parallel PB$ and $DF \parallel PA$ meeting $AB$ produced in $E, F$. $\therefore \triangle PEF = \triangle ABCD$ [see Problem 2.4(ii)]. Divide $EF$ in the given ratio by point $Q$. Hence $PQ$ is the required line.)

92. $ABCD$ is a quadrilateral of which the sides $AB, DC$ meet in $P$ and the sides $AD, BC$ in $Q$. Prove that $PA \cdot PC : PB \cdot PD = QA \cdot QC : QB \cdot QD$. (Draw $BE \parallel AQ$ meeting $PC$ in $E$ and $DF \parallel AP$ meeting $CQ$ in $F$.)

93. Two escribed circles of the triangle $ABC$ are drawn, one touching $AB, AC$ produced in $D, E$ respectively, the other touching $BA, BC$ produced in $F, G$. Through $D$ the diameter $DH$ is drawn. Prove that (a) $AF = BD$; (b) $HC$ produced passes through $F$. (Use Problem 4.30.)

94. Construct a triangle given the base, the difference between the base angles, and the rectangle contained by the sides.

95. Divide a triangle by a straight line drawn through a given point into (a) two equal parts; (b) two parts of which the areas have a given ratio. [(a) Let $D$ be the given point inside $\triangle ABC$. Join $AD$ and make $AE$ subtend $\angle BAE = \angle CAD$. Take $AE$ such that $AD \cdot AE = \frac{1}{2} AB \cdot AC$. Join $DE$ and construct on it an arc of $\odot$ subtending $\angle = \angle EAC$. If this arc cuts $AB$ in $F, F'$, then two solutions are possible by joining $FD, F'D$.]

96. $ABC$ is a triangle whose sides $AB, BC, CA$ are in ratio of $4 : 5 : 6$. Show that $\angle B = 2 \angle C$.

97. $ABC$ is a triangle inscribed in a circle. If through $A$ another circle is described cutting the first one, $AB, AC$ in $E, D, G$, show that $AB : AC = EB + EG : EC + ED$. (Join $AE$ and draw $CM \parallel AE$.)

98. Construct an isosceles triangle having given its vertex angle and the area.

99. $ABC$ is a right-angled triangle with the right angle at $A$. If $AD, AE$ are the altitude and bisector of the right angle and $F$ is the middle point of $BC$, show that $(AB + AC)^2 : AB^2 + AC^2 = FD : FE$.

100. Construct a triangle having given the area and angles. [Let the given area of a triangle be $a^2$. Construct any triangle $AB'C'$ similar to the required triangle, the angles being given. Produce $AB'$ to $D'$ so that $B'D' = \frac{1}{2} CD'$ the altitude of $\triangle AB'C'$. On $AD'$ as diameter, describe a semi-circle which is cut by perpendicular $B'E$ to $AD'$ in $E$. Produce $B'E$ to $F$ such that $B'F = a$ (side of square equal in area to triangle). $AE$ produced meets $FR$ which is $\perp B'F$ in $R$. $RB$ is $\perp AD'$ produced and $BC \parallel B'C'$. Hence $ABC$ is the required triangle.]

101. $ABC$ is an equilateral triangle and $D$ any point in $BC$. Show that if $AL$ be drawn perpendicular to $BC$ and $BM, CN$ perpendicular to $AD$, then $AL^2 = BM^2 + CN^2 + BM \cdot CN$. 


102. If a point inside a triangle \(ABC\). Lines are drawn from the middle points of \(BC\), \(CA\), \(AB\) parallel to \(OA\), \(OB\), \(OC\) respectively. Prove that they meet in a point \(O'\) and that, whatever be the position of \(O\), \(OO'\) passes through a fixed point and is divided by it in the ratio of 2 : 1.

103. In Fig. 59, prove that \(EK\) produced passes through one of the points of trisection of \(BH\).

104. \(H_1, H_2, H_3, H_4\) and \(G_1, G_2, G_3, G_4\) are the orthocenters and centroids of the four triangles \(BCD, ACD, ABD, ABC\) formed by the cyclic quadrilateral \(ABCD\). Show that the two figures \(ABCD\) and \(H_1H_2H_3H_4\) are congruent, also that \(G_1G_2G_3G_4\) and \(H_1H_2H_3H_4\) are similar.

105. Two equal circles having centers \(A, B\) touch at \(C\). A point \(D\), in \(AB\) produced, is the center of a third circle passing through \(C\). Take a common tangent (other than that at \(C\)) to the circles whose centers are \(A\) and \(D\), and let \(P\) and \(Q\) be the points of contact. Draw the line \(CQ\) cutting the circle with center \(B\) in \(M\) and produce it to meet in \(N\) the tangent to this circle at \(E\) which is diametrically opposite to \(C\). Show that \(EN = PQ\) and \(CM = QN\).

106. If two semi-circles are on opposite sides of the same straight line, and the radius of the greater is the diameter of the less, draw the greatest straight line perpendicular to the diameter and terminated by the circles. (Let \(ADB, AEC\) be the semi-circles, \(AC = CB\). Bisect \(AC\) in \(F\) and trisect \(FC\) in \(G\) so that \(GC = 2FG\). Draw \(DGE \bot AC\). Hence \(DE\) is the greatest line.)

107. \(ABC, A'B'C'\) are two similar triangles. In \(BC, CA, AB\) points \(D, E, F\) are taken and \(B'C', C'A', A'B'\) are divided in \(D', E', F'\) similarly to \(BC, CA, AB\) respectively so that \(BD : DC = B'D' : D'C'\), etc. Prove that the triangles \(DEF, D'E'F'\) are similar and that if the straight lines drawn through \(D, E, F\) at right angles to \(BC, CA, AB\) respectively are concurrent, so also are the corresponding straight lines drawn from \(D', E', F'\).

108. \(ABCD\) is a quadrilateral of which the angles \(A\) and \(B\) are right angles. A point \(L\) is taken in \(AB\) such that \(AL : LB = AD : BC\). Show that \(LD \cdot DC + LP^2 = 2AD \cdot BC + PD^2\), where \(P\) is the middle point of \(DC\).

109. Two circles touch each other internally at \(O\) and a chord \(ABCD\) is drawn. The tangent at \(A\) intersects the tangents at \(B\) and \(C\) in \(G, E\). The tangent at \(D\) intersects the tangents at \(B\) and \(C\) in \(F, H\). Prove that \(OA\) bisects the angle \(GOE\) and that \(EFGH\) can be inscribed in a circle which touches both given circles at \(O\).

110. Construct a circle which will cut three straight lines at given angles. (Let \(AB, BC, CA\) forming \(\triangle ABC\) be the given lines. With any point \(O\) as center describe a circle. Draw radius \(OM \perp\) the direction of \(BC\); make \(\angle MOD = \angle MOE = \angle\) angle at which the required circle is to cut \(BC\). \(\therefore DE\) is \(\perp OM\), \(\therefore DE\) is \(\parallel BE\) and \(\therefore\) the angle which tangent at \(D\)
makes with $DE = \frac{1}{2} \angle DOE$. \therefore \odot DME cuts $DE$ at the same angle at which the required circle is to cut $BC$. Similar chords $FG$, $HK$ of circle $DME$ may be found $\parallel BA$, $AC$ and cutting circle $DME$ at the same given angles.)
CHAPTER 5

LOCI AND TRANSVERSALS

Definitions and Theorems

Loci

Definition: If any and every point on a line, part of a line, or group of lines whether straight or curved satisfy an assigned condition and no other point does so, then that line, part of a line, or group of lines is called the locus of the point satisfying that condition.

Among the most important loci are:

5.1. The locus of a point at a given distance from a given point is the circumference of the circle having the given point as center and the given distance as radius.

5.2. The locus of a point at a given distance from a given straight line is a pair of straight lines parallel to the given line, one on each side of it.

5.3. The locus of a point equidistant from two given points is the straight line bisecting at right angles the line joining the given points.

5.4. The locus of a point equidistant from two given intersecting lines is the two bisectors of the angles formed by the two given lines.

5.5. The locus of a point from which tangents of given length or subtending a given angle are drawn to a given circle is another circle concentric with the given one.

5.6. The locus of a point which subtends a given angle at a given line is an arc of a circle passing through the ends of the given line. If this is a right angle then the locus will be the circle on the given line as diameter.

Some of the most frequently used loci are:

5.7. The locus of the mid-points of the chords of given length drawn in a given circle is a circle concentric with the given one and touching the chords at these points.

5.8. The locus of a point the sum of the squares of whose distances from two given points is constant is a circle with the mid-point between the given points as center.

5.9. The locus of a point the difference of the squares of whose distances from two given points is constant is a straight line perpendicular to the line joining the two given points.
5.10. The locus of a point the ratio of whose distances from two fixed points is constant is a circle, called the Apollonius circle.

**Transversals**

5.11. Ceva's theorem: Any three straight lines drawn through the vertices of a triangle so as to intersect in the same point either inside or outside the triangle divide the sides into segments such that the product of three non-adjacent segments is equal to the product of the other three.

5.12. Conversely, if three straight lines drawn through the vertices of a triangle cut the sides themselves, or one side and the other two produced, so that the product of three non-adjacent segments is equal to the product of the others, the three lines are concurrent.

5.13. Menelaus' theorem: If the sides or sides produced of a triangle be cut by a transversal, the product of three non-adjacent segments is equal to the product of the other three.

5.14. Conversely, if three points be taken on two sides and a side produced of a triangle, or on all three sides produced, such that the product of three non-adjacent segments is equal to the product of the others, the three points are collinear.

**Solved Problems**

5.1. Find the locus of a point the sum of whose distances from two given intersecting straight lines is equal to a given length.

**Analysis:** Let $AOB$, $COD$ be the given intersecting straight lines and $X$ the given length (Fig. 131). Between $OA$, $OC$ place the straight line $GK = X$ and $\perp$ to $OA$. This is done by taking $EN = X$ on any line $LN \perp OA$ and drawing $EG \parallel OA$. Then $GK = EN = X$. Cut off
OH = OG and join GH. Hence in the isosceles \( \triangle OGH \), the sum of the \( \perp \) distances of any point in \( GH \) from \( OA, OC \) is equal to \( GK \) (see Problem 1.20). Therefore, every point in \( GH \) satisfies the required condition.

Also, no point within the angle \( AOC \) not in \( GH \) has the sum of its distances from \( OA, OC \) equal to \( X \).

**Proof:** Take any such point \( P \). Through \( P \) draw \( LPM \parallel GH \); draw \( LN \perp AO \). Then the sum of the \( \perp \) distances of \( P \) to \( OA, OC = LN \). But \( LN \) is not equal to \( GK \) or \( X \), since \( LNKG \) cannot be a rectangle because \( NK \) meets \( LG \) and \( O \). And if we take \( OR, OQ \) each equal to \( OG \) and join \( GR, RQ, QH \), it can be shown in the same way that every point on the lines \( GR, RQ, QH \) (and no other) has the sum of its \( \perp \) distances from \( AB, CD \) equal to \( X \). Therefore the perimeter of the rectangle \( GHQR \) is the required locus.

5.2. \( ABCD \) is a quadrilateral and \( P \) is a point inside it such that the sum of the squares on \( PA, PB, PC, PD \) is constant. Show that the locus of \( P \) is a circle and find its center.

**Construction:** Bisect \( AB, BC, CD, DA \) in \( E, F, G, H \) respectively. Join \( EG, FH \) to intersect in \( O \). Then \( O \) is the center of the locus \( \bigcirc \) of \( P \) whose radius is \( OP \) (Fig. 132).

![Figure 132](image)

**Proof:** Join \( PE, PG, OA, OB, OC, OD \). \( O \) is the mid-point of \( EG, FH \) (see Problem 1.14). \( AP^2 + PB^2 = 2(PE^2 + AE^2) \) and \( PC^2 + PD^2 = 2(PG^2 + DG^2) \). \( \therefore AP^2 + PB^2 + PC^2 + PD^2 = 2(PE^2 + PG^2 + AE^2 + DG^2) = 4(PO^2 + OG^2) + 2(AE^2 + DG^2) = 4PO^2 + OA^2 + OB^2 + OC^2 + OD^2 = \) constant. Since \( O \) is the mid-point of \( EG, FH \) and hence fixed and \( OA, OB, OC, OD \) are fixed lengths, \( \therefore PO \) is fixed in length. \( \therefore \) Locus of \( P \) is \( \bigcirc \) with \( O \) as center and \( PO \) as radius.
5.3. From any point $P$ on the circumference of a circle circumscribing a triangle $ABC$ perpendiculars $PD, PE$ are let fall on the sides $AB, BC$. Prove that the locus of the center of the circle circumscribing the triangle $PDE$ is a circle.

**Construction:** Join $PB$ and bisect it in $M$, which will be the center of $\odot$ circumscribing $\triangle PDE$. Join $OB$ and with $OB$ as diameter draw a $\odot$ which is the required locus of $M$ (Fig. 133).

**Figure 133**

**Proof:** As will be seen later, $DE$ is the Simson line of $P$ with respect to $\triangle ABC$. Since $\angle PDB + \angle PEB = 2$ right angles, $\therefore$ quadrangle $PDBE$ is cyclic. Hence $\odot$ circumscribing $\triangle PDE$ will pass through $B$. $\therefore$ $PB$ is a diameter of $\odot PDE$. Since the $\perp$ from $O$ the circumcenter of $\triangle ABC$ bisects $PB$, $\therefore OM$ is $\perp PB$. $\therefore OB$ is a radius of $\odot ABC$ and fixed in position and length, $\therefore$ the locus of $M$, the circumcenter of $\triangle PDE$, is a circle with $OB$ as diameter.

5.4. $M$ and $N$ are the centers of two circles which intersect each other orthogonally at $A, B$. Through $A$ a common chord $CAD$ is drawn to the circles $M, N$ meeting them in $C, D$ respectively. Find the locus of the middle point of $CD$.

**Construction:** Join $MN$ and on it as diameter construct a $\odot$ which will be the locus of $E$ the mid-point of $CAD$. Draw $MG, NK \perp s$ to $CAD$ and join $ME, EN, MA, AN$ (Fig. 134).

**Proof:** Since $\odots$ cut orthogonally, $\therefore MA$ is $\perp AN$. $\therefore MN^2 = MA^2 + AN^2$. $\therefore E, G, K$ are the mid-points of $CAD, CA, AD, \therefore CG + GE = AE + AD$. $\therefore 2GE + AE = AE + 2AK$. $\therefore GE = AK$. Again, $MA^2 + AN^2 = MG^2 + AG^2 + KN^2 + AK^2 = (MG^2 + GE^2) + (KE^2 + KN^2) = EM^2 + EN^2 = MN^2$. $\therefore MEN$ is a right
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angle. \( MN \) is fixed, hence \( O \) on \( MN \) as a diameter is the locus of \( E \) for all locations of \( CAD \).

5.5. \( PQ \) is a chord in a fixed circle such that the sum of the squares on the tangents from \( P, Q \) to another fixed circle is always constant. Show that the locus of \( R \) the middle point of \( PQ \) is a straight line.

**Construction:** Let \( M \) be the center of the \( O \), where \( PQ \) is a chord and \( PC, QD \) are the tangents to another \( O \) center \( N \). From \( R \) draw \( RS \perp \) to \( MN \) and this will be the locus of \( R \). Join \( MP, MR, NP, NQ, NC, ND, NR \) (Fig. 135).

**Proof:** \( PN^2 + NQ^2 = PC^2 + CN^2 + QD^2 + DN^2 \). But \( (PC^2 + QD^2) \) is constant and \( CN = DN = \) fixed radii. \( \therefore PN^2 + NQ^2 = \) constant \( = 2 NR^2 + 2 PR^2 = 2 NR^2 + 2 PM^2 - 2 RM^2 \). But \( PM \) is fixed also. \( \therefore NR^2 - RM^2 \) \( = \) constant. Since \( M, N \) are fixed centers, \( \therefore \) locus of \( R \) is the \( \perp RS \) on \( MN \) (see Problem 5.9).
5.6. If the rectangle $ABCD$ can rotate about the fixed corner $A$ such that $B, D$ move along the circumference of a given circle whose center is $O$, find the locus of the remaining corner $C$.

**Construction:** Join the diagonals $AC, BD$ to intersect in $E$. Then the locus of $C$ is a concentric circle with $O$ as center and $OC$ as radius. Join $OA, OD, OE$ (Fig. 136).

![Figure 136](image)

**Proof:** Since $E$ is the mid-point of $AC, BD$, and $OE$ is $\perp BD$, \[ AO^2 + OC^2 = 2 OE^2 + 2 AE^2 = 2 OE^2 + 2 DE^2 = 2 DO^2. \]

\[ OC^2 = 2 DO^2 - AO^2 = \text{constant} \quad (\text{since } DO, AO \text{ are fixed}). \]

\[ O \text{ is a fixed center, } \therefore \text{the locus of } C \text{ is a concentric circle with center } O \text{ and radius } \sqrt{2 DO^2 - AO^2}. \]

5.7. A circle of constant magnitude passes through a fixed point $A$ and intersects two fixed straight lines $AB, AC$ in $B, C$. Prove that the locus of the orthocenter of the triangle $ABC$ is a circle.

**Construction:** Let $H$ be the orthocenter of $\triangle ABC$, and $O$ be the center of $\odot ABC$. Join $OC$ and drop $OD \perp BC$. With $A$ as center and $AH$ as radius construct a circle to be the locus of $H$ (Fig. 137).

**Proof:** Since the $\odot O$ is of constant magnitude and $A, AF, AL$ are fixed in position, then $BC$ is of constant magnitude. \[ \therefore OD \perp BC, \]

\[ \therefore D \text{ is the mid-point of } BC. \]

\[ \therefore \angle DOC = \angle BAC. \]

Since $OD = CD \cdot \cot DOC = CD \cdot \cot BAC$ and $OD = \frac{1}{2} AH$ (see Problem 1.32), \[ \therefore AH = 2 CD \cdot \cot A = BC \cdot \cot A. \]

\[ \therefore BC \text{ is fixed in length and } \angle A \text{ is constant, } \therefore AH \text{ is a fixed length. But, since } A \text{ is a fixed point, the locus of } H \text{ is a circle with } A \text{ as center and } (BC \cdot \cot A) \text{ as radius.} \]

5.8. Through a fixed point $O$ any straight line $OPQ$ is drawn cutting a fixed circle $M$ in $P$ and $Q$. On $OP, OQ$ as chords are described two circles touching
the fixed circle at $P, Q$. Prove that the two circles so described intersect on another fixed circle.

**Construction:** Let this point of intersection be $S$. Join $OS, PS, QS$ and draw $PR, QR$ tangents to the $\odot M$. Join $RM$ and $OM$. On $OM$ as diameter draw a $\odot$ which will be the locus of $S$ (Fig. 138).

**Proof:** In $\triangle SOQ$, $\angle SOQ + \angle SQO + \angle OSQ = 2$ right angles. \(\therefore\) $RQ$ touches $\odot N$ passing through $\triangle OSQ$, \(\therefore\) $\angle RQS = \angle SOQ$. \(\therefore\)
\[ \angle RQO + \angle OSQ = 2 \text{ right angles.} \] Since \[ \angle RQS = \angle RPS = \angle POS, \] \[ \therefore \text{SPQR is a cyclic } \triangle. \] But \[ \text{PMQR is also cyclic. Then the figure SPMQR is cyclic. Hence } \angle MSQ = \angle MRQ. \] \[ \therefore \angle RQO + \angle MRQ + \angle MSO = 2 \text{ right angles.} \] \[ \therefore \angle RQO + \angle MRQ = \text{right angle (RM \perp PQ)}, \therefore \angle MSO = \text{right angle. But } O, M \text{ are two fixed points. Therefore, } \odot \text{ drawn on } OM \text{ as diameter is the locus of } S \text{ and is a fixed } \odot.

5.9. Given a fixed circle and two fixed points \( A, B. \) From \( A, \) a line \( AC \) is drawn to intersect the circle in \( C. \) Produce \( AC \) to \( D \) so that \( AC = CD. \) \( E \) is the middle point of \( AB. \) If \( CB \) intersects \( DE \) in \( M, \) find the locus of \( M. \)

**CONSTRUCTION:** Let \( O \) be the center of the fixed \( \odot. \) Join \( DB, OB, OC. \) Draw \( MN \parallel CO \) to cut \( OB \) in \( N. \) Then the locus of \( M \) is a circle with center \( N \) and radius \( MN \) (Fig. 139).

**Proof:** \( \therefore C, E \) are the mid-points of \( AD, AB \) in \( \triangle ADB. \) \( \therefore M \) is the centroid of \( \triangle ADB. \) \( \therefore BM = \frac{2}{3} BC. \) \( \therefore MN \parallel CO \) in \( \triangle BOC, \therefore BM/BC = MN/OC = \frac{2}{3}. \) \( \therefore MN = \frac{2}{3} OC, \) the fixed radius of \( \odot O. \) Hence \( MN \) is of fixed length. Again, \( BM/BC = BN/BO = \frac{2}{3}. \therefore BN = \frac{2}{3} BO. \therefore B \text{ and } O \text{ are both fixed and } BN \text{ is of fixed length. } \therefore N \text{ is a fixed point on } BO. \text{ Hence locus of } M \text{ is a } \odot \text{ with center } N \text{ and radius } MN = \frac{2}{3} \text{ radius } OC.

**Note:** If, in general, \( E \) divides \( AB \) into \( EB/AE = \) given ratio \( p \) and \( C \) divides \( AD \) into \( CD/AD = \) given ratio \( q, \therefore \) in \( \triangle ABC: (CD/DA) (AE/EB)(BM/MC) = 1 \) (Th. 5.13). \( \therefore BM/MC = (DA/CD)\)
\[(EB/AE) = p/q. \quad \therefore BM/BC = p/(p + q) = MN/CO. \quad \therefore MN = CO \left(\frac{p}{p + q}\right)\] and \(N\) is also fixed on \(OB\). \(\therefore\) Locus of \(M\) is \(\odot N\) with \(MN\) as radius.

5.10. The vertices of a triangle are on three straight lines which diverge from a point, and the sides are in fixed directions; find the locus of the center of the circumscribed circle.

**Construction:** Let \(DEF, D'E'F'\) be two \(\Delta\)s with their vertices on \(OD', OEE', OFF'\). Bisect \(DF\) in \(G\) and produce \(OG\) to meet \(D'F'\) in \(G'\). Let \(C, C'\) be the centers of the circumscribing \(\odot\)s on \(DEF, D'E'F'\). Then \(OC, OC'\) will be the locus of \(C\) (Fig. 140).

**Figure 140**

**Proof:** Since \(G\) is the mid-point of \(DF\) and \(D'F'\) is \(\parallel DF\), \(\therefore G'\) bisects \(D'F'\). \(\therefore \angle DEF = \angle D'E'F', \therefore \angle DCF = \angle D'C'F'. \therefore \angle GCF = \angle G'C'F'\) and right angle \(CGF\) = right angle \(C'G'F'\). \(\therefore CG/C'G' = FG/F'G' = OG/OG'\) and \(\angle CGO = \angle C'G'O. \therefore \angle GOC = \angle G'OC'. \therefore OCC'\) is a straight line and is the required locus.

**Note:** This is an explicit proof of a relationship that can also be developed by homothetic figures.

5.11. Find the locus of a point moving inside an equilateral triangle such that the sum of the squares of its distances from the vertices of the triangle is constant.

**Construction:** Let \(O\) be the center of the \(\odot\) circumscribing the equilateral. \(\triangle ABC\) and \(P\) is a point which satisfies the condition. The locus of \(P\) will be a \(\odot\) with center \(O\) and \(OP\) as radius (Fig. 141).
Proof: Join $AO, BO, CO$ and draw $DOEF \perp OP$. From $A, B, C$ the \( \angle AD, BE, CF \) are drawn on $DOEF$. Now, $PB^2 = BO^2 + OP^2 - 2\ PO \cdot BE$. $PA^2 = AO^2 + OP^2 + 2\ PO \cdot AD$. $PC^2 = CO^2 + OP^2 + 2\ PO \cdot CF$. Since $AO = BO = CO = r$, then by adding, $PA^2 + PB^2 + PC^2 = 3r^2 + 3\ OP^2 + 2\ PO(AD + CF - BE) = \text{constant}$.\
\[ AD + CF = BE \text{ (see Problem 1.29)}, \therefore \ OP^2 = \frac{1}{3} (\text{construct} - 3r^2) = \text{constant}. \] Hence, the locus of $P$ is a circle with $O$ as center and $OP$ as radius.

5.12. If a triangle $ABC$ is similar to a given triangle and has one vertex $A$ fixed, while another vertex $B$ moves along a given circle, prove that the locus of the third vertex $C$ is a circle.

Construction: Suppose that $M$ is the center of the $\bigcirc$ on which $B$ moves. Join $MA$ and draw the lines $MN, AN$ to make with $MA \angle NMA = \angle B$ and $\angle NAM = \angle CAB$. Then the locus of $C$ is the $\bigcirc$ with $N$ as center and $CN$ as radius (Fig. 142).

Proof: Join $MB$. \( \because \ \triangle AMN, ABC \) are similar, \( \therefore \ \triangle AM/AB = AN/AC. \ \therefore \ \triangle MAB = \triangle NAC, \ \therefore \ \triangle MAB, NAC \) are also similar. Hence $MB/NC = AB/AC = \text{given ratio}$. But, since $MB$ is a given radius, then $NC$ is also given. Again, the $\triangle MNA$ has two of its vertices $M, A$ fixed, and its angles are fixed because it is similar to $\triangle ABC$. \( \therefore \ N \) is a fixed point. Therefore, $\bigcirc N$ with radius $= MB \cdot AC/AB$ is the required locus of $C$.

5.13. Two fixed straight lines $AB, CD$ of given lengths meet, when produced, at a point $O$. $P$ is a point such that the sum or difference of the areas of the triangles with $P$ as vertex and $AB, CD$ as bases is equal to the area of a
given triangle. Prove that P lies on a fixed straight line and construct this line for each case.

**Construction:** (i) For the case of a given sum of $\triangle PAB, PCD$, produce $AB, CD$ to meet in $O$. Take $OQ = AB$ and $OR = CD$, and join $QR$. Then the locus of $P$ will be the line $MPL$ drawn through $P \parallel QR$ (Fig. 143).

**Proof:** Since $\triangle APB = \triangle QPO$ and $\triangle PDC = \triangle PRO$, adding gives $\triangle QRO = \triangle APB + \triangle PDC = \text{constant}$. \(\therefore\) The area of $\triangle OQR$ is fixed (since $AB, CD$ are fixed straight lines and $O$ is fixed), $\triangle PQR$ is constant and since $QR$ is fixed in direction, \(\therefore\) the locus of $P$ is a line through it $MPL \parallel QR$.

**Construction:** (ii) For the case of a given difference of $\triangle PAB, PCD$, produce $ABO$ to $Q$ so that $OQ = AB$ and take $OR$ on $OC$
= CD and join QR. Then the locus of P is the line MPL || QR (Fig. 144).

**Proof:** Similar to the first case. \( \triangle PCD - \triangle APB = \triangle PRO - \triangle POQ = \triangle PRQ = \triangle OQR - \triangle PQR = \triangle OQR \) = constant. But, since \( \triangle OQR \) is fixed, \( \triangle PQR \) is constant and also QR is fixed in direction. \( \therefore \) The locus of P is a line MPL || QR.

5.14. From B, C the vertices of a triangle right-angled at A are drawn straight lines BF, CE respectively parallel to AC, AB and proportional to AB, AC. Find the locus of the intersection of BE and CF.

**Construction:** Let P be a point on the locus. Draw EG, FH, PK \perp BC and FL, PM \perp AC. Let PM meet BF, BC in N, R (Fig. 145).
Proof: $\triangle s$ $ABC$, $HFB$ are similar. \therefore BH/BF = AC/BC. But BF/CE = AB/AC (hypothesis). Hence BH/CE = AB/BC = CG/CE (since $\triangle s$ $ABC$, $GCE$ are also similar). \therefore BH = CG. Also, EG/CG = BH/HF. \therefore EG/HF = EG^2/CG^2 = AC^2/AB^2.

Again, PM/AB = CP/CF = PK/FH and AC/AM = BE/BP = EG/PK. \therefore PM\cdot AC/AB\cdot AM = EG/FH = AC^2/AB^2. \therefore PM/AM = AC/AB. Hence $\triangle s$ $PAM$, $CBA$ are similar (Th. 4.86). \therefore \angle PAM = \angle ABC. \therefore$ The locus of $P$ is a line through $A$ making with $CA$ or $CA$ produced (according as $CE$, $BF$ lie on the same side or on opposite sides of $BC$) an angle $= \angle ABC$ and which coincides with $\perp$ from $A$ to $BC$.

5.15. Four points $A$, $B$, $A'$, $B'$ are given in a plane with $AB$ different from $A'B'$. Prove that there are always two positions of a point $C$ in the plane such that the triangles $CAB$, $CA'B'$ are similar, the equal angles being denoted by corresponding letters.

CONSTRUCTION: Draw $AA'$, $BB'$ and divide them internally and externally in the given ratio ($AB/A'B'$) in $P$, $Q$ and $M$, $N$ respectively. On $PQ$, $MN$ as diameters describe two $\bigcirc$s to intersect in $C$, $C'$ which are the required points (Fig. 146).

![Figure 146](image)

Proof: Join $C$, $C'$ to the four corners $A$, $B$, $A'$, $B'$. Since $\bigcirc$ on $PQ$ as diameter is the Apollonius $\bigcirc$ of $C$ in $\triangle CAA'$, \therefore $AC/A'C = AP/A'P = AB/A'B'$. Similarly, $BC/B'C = AB/A'B'$. Hence $AB/A'B' =$
AC/A'C = BC/B'C. Therefore, ∆s ACB, A'CB' are similar. Similarly, ∆s AC'B, A'C'B' are similar. Therefore, there are two positions of C.

5.16. The hypotenuse of a right-angled triangle is given. Find the locus of the mid-point of the line joining the outer vertices of the equilateral triangles described externally on its sides.

Construction: Let BC be the hypotenuse of right-angled ∆ABC. On CA, AB construct equilateral ∆s CAD, AEB. On the same side as ∆ABC, construct on BC equilateral ∆BFC. Bisect BC, CF, FB, DE in G, H, K, P. Join GP, GH, GK, GD, GE, GA, GF. Therefore, the required locus of P is the circle described on HK as diameter (Fig. 147).

Proof: :: GA = GC and AD = DC, :: GD is ⊥ AC and :: || AB. Similarly, GE is ⊥ AB and || AC. :: ∠GDC = 1/2 ∠ADC = ∠GFC, :: D lies on circle FG whose center is H. :: HD = HG. :: GD is || AB and GE || AC, :: ∠EGD is right. :: GP = PD and PH is common to ∆s GPH, DPH and GH = HD. :: ∠GPH = ∠DPH. Similarly, ∠GPK = ∠KPE. :: ∠KPH is right. Since H, K are fixed points, :: locus of P is a circle on diameter HK.

5.17. The base and the vertex angle opposite to it in a triangle are given. Find the locus of the center of the circle which passes through the excenters of the three circles touching the sides of the triangle externally.

Construction: Let O1, O2, O3 be the centers of the excircles touching BC, CA, AB of ∆ABC in which BC and ∠A are given. Draw the sides of ∆O1, O2, O3 and draw circle circumscribing ∆ABC cutting O01 and O2O3 in D, E (O being the incenter of ∆ABC). Bisect DE in M, and produce OM to meet EP, which is ⊥ O2O3, in P the center of circle circumscribing O1O2O3. Join AOO1, BOO2, COO3 (Fig. 148).
FIGURE 148

Proof: Since $BC$ and $\angle A$ are given, $\therefore \triangle ABC$ is fixed. $\therefore \triangle ABC$ is the pedal $\triangle$ of $\triangle O_1O_2O_3$, the fixed $\odot ABC$ is the nine-point $\odot$ of $\triangle O_1O_2O_3$, and $D, E$ will be the mid-points of $OO_1, O_2O_3$. ($O$ is also the orthocenter of $\triangle O_1O_2O_3$.) Since $\angle DAE = \text{right angle}$, $\therefore DE$ is a diameter of the fixed $\odot ABC$, whose center $M$ is also fixed. But $D$ is a fixed point (being the mid-point of the fixed arc $BC$). $\therefore E$ is also a fixed point. $\therefore OM = MP$ (see Problem 1.32), $\triangle$s $OMD, PME$ are congruent. $\therefore PE = OD$. But $\angle BOC = \text{right} + \frac{1}{2} \angle A$ is fixed and $BC$ is fixed. Then $\odot BOC$ is fixed and since it passes through $O_1$ ($BOCO_1$ is a cyclic quadrilateral) its radius $OD$ is fixed. $\therefore PE$ is fixed in length. Therefore, the locus of $P$ is a $\odot$ with $E$ as center and $PE$ as radius.

5.18. Find the locus of a point such that the sum of the squares on the tangents from it to four given circles may be equal to a given square.

Construction: Let $A, B, C, D$ be four given circles and $G$ a moving point, such that the sum of the squares of the tangents $GH, GJ, GK, GL$ to these $\odot$s is given. Draw $AB, CD$ and bisect them in $P, Q$ respectively. Bisect $PQ$ in $R$ and join $A, B, C, D$ to $G$ (Fig. 149).

Proof: $GA^2 + GB^2 + GC^2 + GD^2 = (GH^2 + GJ^2 + GK^2 + GL^2) + (AH^2 + BJ^2 + CK^2 = DL^2) = \text{constant}$, since the radii are given. Hence $2( GP^2 + PB^2 + GQ^2 + QC^2) = \text{constant}$. $\therefore (PB^2 + QC^2)$ is a fixed quantity, $\therefore GP^2 + GQ^2 = \text{constant} = 2 (GR^2 + PR^2)$. But, since $P, Q$ are two fixed points, $\therefore R$ is fixed and $PR$ is
of constant magnitude. Hence $GR$ is constant. Therefore, the locus of $G$ is a $\circ$ having $R$ as center and $RG$ as radius.

5.19. A line meets the sides $BC, CA, AB$ of a triangle $ABC$ at $D, E, F$. $P, Q, R$ bisect $EF, FD, DE$. $AP, BQ, CR$, produced if necessary, meet $BC, CA, AB$ at $X, Y, Z$. Show that $X, Y, Z$ are collinear (Fig. 150).

\[ \frac{\triangle ABX}{\triangle APF} = \frac{AX \cdot AB \sin BAX}{AP \cdot AF \sin BAX} = \frac{AX \cdot AB}{AP \cdot AF}. \]

\[ \frac{\triangle APE}{\triangle ACX} = \frac{AP \cdot AE}{AC \cdot AX}. \therefore \triangle APF = \triangle APE \text{ (since } PF \]
\( \triangle ABX/\triangle ACX = BX/XC = AB \cdot AE/AC \cdot AF \). Similarly, \( CY/YA = BC \cdot BF/BA \cdot BD \) and \( AZ/ZB = CA \cdot CD/CB \cdot CE \). Multiplying yields

\[
\frac{BX \cdot CY \cdot AZ}{XC \cdot YA \cdot ZB} = \frac{AB \cdot BC \cdot CA \cdot AE \cdot BF \cdot CD}{AC \cdot BA \cdot CB \cdot AF \cdot BD \cdot CE} = 1.
\]

Therefore, \( X, Y, Z \) are collinear (converse, Menelaus' Th. 5.14).

5.20. If \( P, Q, R \) are the points of contact of the inscribed circle with the sides \( BC, CA, AB \) of a triangle \( ABC \) and \( PQ, QR, RP \) produced meet \( AB, BC, CA \) in \( G, H, K \), show that \( GHK \) is a straight line. If \( X, Y, Z \) are the mid-points of \( RG, PH, QK \), show also that these points are collinear (Fig. 151).

![Figure 151](image)

Proof: (i) \( AG \cdot BP \cdot CQ/GB \cdot PC \cdot QA = 1 \) (Th. 5.14). \( \therefore CP = CQ \), \( \therefore AG/GB = QA/BP \). Similarly, \( BH/HC = BR/CQ \) and \( CK/AK = CP/AR \). Multiplying gives \( AG \cdot BH \cdot CQ/GB \cdot HC \cdot KA = 1 \). \( \therefore GHK \) is a straight line.

(ii) Again, consider the line \( HYCPB \) alone. Draw a \( \odot \) on \( HP \) as diameter. Since \( HC/\odot = HC/\odot = CP/BP \), \( \therefore CP/HC = BP/\odot \) and \( CP/HC - CP = BP/\odot - BP \) or \( CP/BP = YC/YP \) \( (1) \) \( (Y \) is the mid-point of \( HP \)). \( \therefore \) The \( \odot \) on \( HP \) as diameter is the Apollonius \( \odot \) of \( CB \) with respect to \( P, H \), \( \therefore \) any point \( E \) on this \( \odot \) will yield the ratio \( EC/EB = CP/BP \). \( \therefore \angle PEH = \text{right angle} \) and \( EP, EH \) are the internal and external bisectors of \( \angle E \). \( \therefore \) \( \triangle YEC, YBE \) are similar.

\( \therefore YE/YB = EC/EB = CP/BP \) \( (2) \). \( \therefore YP = YE \), hence from (1) and
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(2), \[ \text{YC}/YB = CP^2/BP^2. \] Similarly, \[ \text{XB}/XA = BR^2/AR^2. \] \[ AZ/ZC = AQ^2/CQ^2. \] By multiplying, \[ \text{YC} \cdot \text{XB} : \text{AZ} = \text{YB} \cdot \text{XA} : \text{ZC} = 1. \] Hence \( XYZ \) is also a straight line.

5.21. The sides \( BC, CA, AB \) of a triangle \( ABC \) are cut by two lines in the points \( D, E, F \) and \( D', E', F' \). Show that \( EF', FD', DE' \) cut \( BC, CA, AB \) in three collinear points \( D'', E'', F'' \) (Fig. 152).

\[ \text{FIGURE 152} \]

Proof: Since \( EF'D'' \) is a transversal of \( \triangle ABC \), \[ \text{AE} \cdot \text{CD}'' \cdot \text{BF}' / \text{EC} \cdot \text{D}' \cdot \text{F}' \cdot \text{A} = 1. \] Similarly with the other two transversals \( E''FD', DE''F' \):

\[ \text{BF} \cdot \text{AE}'' / \text{FA} \cdot \text{E}'' \cdot \text{CD}' = 1 \quad \text{and} \quad \text{AE}' \cdot \text{CD} \cdot \text{BF}'' / \text{EC}' \cdot \text{DB} \cdot \text{F}' \cdot \text{A} = 1. \]

But, \( DEF, D'E'F' \) are two transversals; \[ \therefore \text{multiplying and using the ratios from the other transversals yields} \]

\[ \text{CD}'' \cdot \text{BF}'' \cdot \text{AE}'' / \text{D}'' \cdot \text{F}'' \cdot \text{A} \cdot \text{E}'' \cdot \text{C} = 1. \]

Therefore, \( D'', E'', F'' \) are collinear.

5.22. A transversal \( DEF \) cuts the sides \( BC, CA, AB \) of a triangle \( ABC \) in \( D, E, F \) respectively. If \( AD, BE, CF \) are joined and \( AG, BL, CH \) are drawn such that \( \angle BAG = \angle CAD, \angle CBE = \angle ABL, \angle ACF = \angle BCH, \) show that \( GHL \) is a straight line (Fig. 153).

Proof:

\[ \frac{\triangle ABG}{\triangle ACD} = \frac{\text{AB} \cdot \text{AG} \sin BAG}{\text{AC} \cdot \text{AD} \sin CAD} = \frac{\text{AB} \cdot \text{AG}}{\text{AC} \cdot \text{AD}} = \frac{\text{BG}}{\text{DC}}. \]

Likewise, \[ \angle ACG / \angle ABD = \frac{\text{AC} \cdot \text{AG} / \text{AB} \cdot \text{AD} = \text{CG} / \text{BD}}. \] Dividing gives \( \text{BD} \cdot \text{BG} / \text{DC} \cdot \text{CG} = \text{AB}^2 / \text{AC}^2. \) Similarly, \( \text{CE} \cdot \text{CL} / \text{AE} \cdot \text{AL} = \)
$BC^2/AB^2$ and $AF \cdot AH/BF \cdot BH = AC^2/BC^2$, since $DEF$ is a transversal. 
'. Multiplying and using the ratios from the other transversals gives $BG \cdot CL \cdot AH/GC \cdot LA \cdot HB = 1$. Hence $GHL$ is a straight line.

5.23. $AB$, $CD$, $EF$ are three parallel straight lines. $M$, $N$, $R$ are the intersections of pairs of lines $AD$, $BC$; $AF$, $BE$; $CF$, $DE$ respectively. If $X$, $Y$, $Z$ are the middle points of $AB$, $CD$, $EF$ respectively, show that $XR$, $YN$, $ZM$ are concurrent.

**Construction:** Draw $XR$, $YN$, $ZM$, and also $XY$, $YZ$, $ZX$ (Fig. 154).

**Proof:** '. $AB$ is $\parallel CD$ and $X$ is the mid-point of $AB$, '. $XM$ produced bisects $CD$. Hence $XMY$ is one straight line. Likewise, $YRZ$, $ZNX$ are also straight lines. By similarity, $XM/MY = AX/DY$, $YR/RZ = DY/ZE$, and $ZN/NX = ZF/AX$. Multiplying gives $XM \cdot YR \cdot ZN/MY \cdot RZ \cdot NX = 1$. Therefore, $XR$, $YN$, $ZM$ are concurrent (Th. 5.12).
5.24. On the sides BC, CA, AB of a triangle are taken the points X, Y, Z such that BX = XC, CY = YA, AZ = 2 ZB. BY and CZ meet at P, AX and CZ at Q, AP and BQ at R, BP and CR at S. Show that BY = 6 SP.

Construction: Produce AR, BQ, CS to meet BC, CA, AB in D, E, F. Join DS, ZS (Fig. 155).

\[
\text{Proof: } \because AE \cdot CX \cdot BZ / EC \cdot XB \cdot ZA = 1, \text{ and since } CX = XB, \therefore AE / EC = ZA / BZ = 2. \text{ In } \triangle ACZ, ZB \cdot AY \cdot CP / BA \cdot CY \cdot PZ = 1. \therefore AY = CY, \therefore ZB / BA = PZ / CP = 1 : 3. \text{ Also,}
\]

\[
\frac{AY \cdot CD \cdot BZ}{YC \cdot DB \cdot ZA} = 1.
\]

\[
\therefore CD / DB = ZA / BZ = 2. \text{ Similarly, } AE \cdot CD \cdot BF / EC \cdot DB \cdot FA = 1. \therefore BF / FA = EC \cdot DB / AE \cdot CD = 1 : 4. \text{ Hence } BF / AB = 1 : 5 \text{ and } \therefore AB / BZ = 3, \therefore BF / BZ = 3 : 5. \therefore BF / FZ = 3 : 2. \text{ Therefore,}
\]

\[
ZP \cdot CD \cdot BF / PC \cdot DB \cdot FZ = 1. \therefore BP, CF, DZ \text{ are concurrent at } S. \text{ Hence } ZSD \text{ is one straight line } \parallel AC. \therefore SP / PY = ZP / PC = 1 : 3. \therefore SY / BY = AZ / AB = 2 : 3, \therefore SP / BY = 1 : 6.
\]

5.25. If a transversal cuts the sides BC, CA, AB of a triangle ABC in P, Q, R respectively and if P', Q', R' are the harmonic conjugates of P, Q, R with respect to B, C; C, A; A, B respectively, then AP', BQ', CR' are concurrent. If X, Y, Z be the points of bisection of PP', QQ', RR', then XYZ is a straight line.

Note: Since P', Q', R' are the harmonic conjugates of P, Q, R with respect to B, C; C, A and A, B, then P, P' divide BC in the same ratio; i.e., BP / CP = BP' / CP', and so on.

Construction: Draw PR', R'Q', P'Q, R'Q, P'R, RQ' (Fig. 156).

Proof: CP \cdot BR \cdot AQ / PB \cdot RA \cdot QC = 1. Replacing equal ratios in the
above quantity, \( CP' \cdot BR' \cdot AQ'/P'B \cdot R'A \cdot Q'C = 1 \). Hence \( AP' \), \( BQ' \), \( CR' \) are concurrent (Th. 5.12). Also, \( CP' \cdot BR' \cdot AQ/P'B \cdot R'A \cdot QC = 1 \). Therefore, \( P'QR' \) is a straight line (Th. 5.14). Likewise, \( PR'Q' \) and \( P'RQ' \) are straight lines. Now, in the complete quadrilateral \( QRQ'R' \), \( X \), \( Y \), \( Z \) are the mid-points of its diagonals \( PP' \), \( QQ' \), \( RR' \). Hence they are collinear (see Problem 2.16).

5.26. If \( AD \), \( BE \), \( CF \) are three concurrent lines drawn from the vertices of the triangle \( ABC \) and terminated by the opposite sides, then the diameter of the circle circumscribing triangle \( ABC \) will be equal to \( (AF \cdot BD \cdot CE) : \triangle DEF \).

Construction: Draw the \( \perp \)s \( BL \), \( DR \), \( CM \) to \( EF \) or produced; \( BN \), \( CH \) to \( DR \) and \( AG \) to \( BC \) (Fig. 157).

Proof: Let \( 2r \) be the diameter of the circumscribing \( \odot ABC \). \( \therefore \) \( BD/CD = ND/DH = (DR - LB)/(CM - DR) \). \( \therefore \) \( BD \cdot CM + CD \cdot BL = DR (BD + CD) = BC \cdot DR \). Multiplying both sides by \( EF \) yields \( \therefore \) \( BD \cdot \triangle CEF + CD \cdot \triangle BEF = BC \cdot \triangle DEF \). Hence \( \triangle DEF = (BD \cdot \triangle CEF + CD \cdot \triangle BEF) : BC \). But \( \triangle CEF \cdot \triangle ACF = CE \cdot AC \) and \( \triangle ACF \cdot \triangle ABC = AF \cdot AB \). \( \therefore \) \( \triangle CEF \cdot \triangle ABC = CE \cdot AF \cdot AC \cdot AB \). Similarly, \( \triangle BEF \cdot \triangle ABC = BF \cdot AE \cdot AB \cdot AC \). Substituting gives \( \triangle DEF = (BD \cdot CE \cdot AF \cdot \triangle ABC + CD \cdot BF \cdot AE \cdot \triangle ABC) : BC \cdot AC \cdot AB \). But \( BD \cdot CE \cdot AF = CD \cdot BF \cdot AE \) (Th. 5.11) and \( AB \cdot AC = AG \cdot 2r \) (Th. 4.100). Hence \( BC \cdot AC \cdot AB = 2r \cdot BC \cdot AG = 4r \cdot \triangle ABC \). Therefore, \( \triangle DEF = 2 \cdot AF \cdot BD \cdot CE \cdot \triangle ABC / 4r \cdot \triangle ABC \) or \( 2r = AF \cdot BD \cdot CE / \triangle DEF \).
5.27. $D$, $E$, $F$ are the points of contact of the escribed circle opposite the vertex $A$ with the sides $BC$, $CA$, $AB$ of a triangle $ABC$. $X$, $Y$, $Z$ and $L$, $M$, $N$ are the similar points of contact of the other two escribed circles opposite to $B$, $C$ with the same order of sides. If $BY$, $CN$ intersect in $G$ and $BE$, $CF$ in $H$, show that $A$, $G$, $D$, $H$ are collinear. If $YN$, $ND$, $YD$ are also produced to meet $BC$, $AC$, $AB$ in $P$, $Q$, $R$, then $PQR$ is a straight line (Fig. 158).
Proof: Let the perimeter of \( \triangle ABC = 2s \) and \( a, b, c \) denote the sides \( BC, CA, AB \). \( BD = BF = AF - AB = s - c \). But \( \frac{BD \cdot CY \cdot AN}{DC \cdot YA \cdot NB} = \frac{(s - c) \cdot (s - a) \cdot (s - b)}{(s - b) \cdot (s - c) \cdot (s - a)} = 1 \). Hence \( AD, BY, CN \) are concurrent or \( A, G, D \) are collinear. Also, \( \frac{BD \cdot CE \cdot AF}{DC \cdot EA \cdot FB} = \frac{(s - c) \cdot (s - b) \cdot s}{(s - b) \cdot s \cdot (s - c)} = 1 \). Thus \( H \) lies on \( AGD \) produced. Since \( YNP \) is a transversal of \( \triangle ABC \), \( \frac{BP}{PC} = \frac{(s - c)}{(s - b)} \). Similarly, \( \frac{CQ}{QA} = \frac{(s - a)}{(s - c)} \) and \( \frac{AR}{RB} = \frac{(s - b)}{(s - a)} \). Hence \( \frac{BP \cdot CQ \cdot AR}{PC \cdot QA \cdot RB} = 1 \). Therefore, \( PQR \) is a straight line.

5.28. In the triangle \( ABC \), \( AD, BE, CF \) are the altitudes. If \( EF, FD, DE \) produced meet \( BC, CA, AB \) respectively in \( X, Y, Z \), show that the centers of the circles \( ADX, BEY, CFZ \) are collinear.

Construction: Let \( G, H, J \) be the centers of the circles \( ADX, BEY, CFZ \). Join \( XZ, YZ \) (Fig. 159).

\[ \text{Figure 159} \]

Proof: \( \therefore \frac{BD \cdot CE \cdot AF}{DC \cdot EA \cdot FB} = 1 \) and \( \frac{BX \cdot CE \cdot AF}{XC \cdot EA \cdot FB} = 1 \), hence \( \frac{BD}{DC} = \frac{BX}{XC} \). Likewise, \( \frac{CY}{YA} = \frac{CE}{EA} \) and \( \frac{AZ}{ZB} = \frac{AF}{FB} \). Multiplying yields \( \frac{BX \cdot CY \cdot AZ}{XC \cdot YA \cdot ZB} = 1 \). Therefore, \( XYZ \) is a straight line, since \( \angle ADX \) = right. \( \therefore G \) is the mid-point of \( AX \) the diameter of \( \odot ADX \). Similarly, \( H, J \) are the mid-points of \( BY, CZ \). But, since \( G, H, J \) are
the mid-points of the diagonals of the complete quadrilateral $ACXZ$, then they are collinear (see Problem 2.16).

**5.29.** On the sides $BC, CA, AB$ of a triangle are taken points $X, X'$ and $Y, Y'$ and $Z, Z'$ such that $X', Y, Z$ are collinear and also $X, Y', Z$ and also $X, Y, Z'$. If $X'Z'$ cuts $BY$ in $Q$ and $X'Y'$ cuts $CZ$ in $R$, show that $QY'$ and $RZ'$ meet on $BC$ (Fig. 160).

**Proof:** Produce $QY'$ to meet $BC$ in $P$. $Z'YX$ is a transversal to $\triangle BQX'$: \[\frac{QZ'}{QB} \cdot \frac{BY}{BY'} \cdot \frac{X'X}{X'Y} \cdot \frac{YQ}{YX} \cdot \frac{XB}{XB'} = 1.\] Likewise, in $\triangle YQY'$ and $\triangle YY'X'$, \[\frac{Y'P}{Y'B} \cdot \frac{YC}{Y'C} \cdot \frac{QB}{PQ} \cdot \frac{CY'}{CY} \cdot \frac{BY}{BY'} = 1, \quad \text{and} \quad \frac{X'R}{X'B} \cdot \frac{Y'C}{Y'C} \cdot \frac{YZ}{YZ'} \cdot \frac{RY'}{RY} \cdot \frac{ZX'}{ZX} = 1.\]

But, since in $\triangle YXX'$ \[\frac{YZ}{XX'} \cdot \frac{YQ}{YX} \cdot \frac{X'B}{XB} \cdot \frac{YQ}{YX} \cdot \frac{ZX'}{ZX} = 1.\] Therefore, $Z'R$ is a straight line.

**5.30.** A circle meets the sides $BC, CA, AB$ of a triangle $ABC$ at $A_1, A_2; B_1, B_2; C_1, C_2$. $B_1C_1, B_2C_2$ meet at $X; C_1A_1, C_2A_2$ at $Y$; and $A_1B_1, A_2B_2$ at $Z$. Show that $AX, BY, CZ$ are concurrent.

**Construction:** Join $A_1B_2, B_1C_2, C_1A_2$ and produce $AX$ to meet $B_1C_1$ in $D$ (Fig. 161).
Proof: Since $AX$, $B_1C_1$, $B_2C_2$ are concurrent, \( \therefore \) $DC_2 \cdot B_1B_2 \cdot AC_1/DB_1 \cdot AB_2 \cdot C_1C_2 = 1$, since

\[
\frac{DC_2}{DB_1} = \frac{\triangle ADC_2}{\triangle ADB_1} = \frac{AC_2 \cdot AD \cdot \sin C_2AD}{AD \cdot AB \cdot \sin DAB_1} = \frac{AC_2 \cdot \sin C_2AD}{AB_1 \cdot \sin DAB_1},
\]

and so on. Now substituting in the above quantity and simplifying,

\[
\frac{\sin C_2AD \cdot \sin B_2B_1C_1 \cdot \sin B_1C_2B_2}{\sin DAB_1 \cdot \sin C_1B_1C_2 \cdot \sin B_2C_2C_1} = 1.
\]

\[
\frac{\sin BAX}{\sin XAC} = \frac{\sin C_1B_1C_2 \cdot \sin B_2C_2C_1}{\sin B_2B_1C_1 \cdot \sin B_1C_2C_2} \quad \vdots
\]

and so on. Hence

\[
(sin BAX/sin XAC) \cdot (sin CBY/sin YBA) \cdot (sin ACZ/sin ZCB)
\]

\[
= \frac{\sin C_1B_1C_2 \cdot \sin B_2C_2C_1 \cdot \sin A_1C_1A_2 \cdot \sin A_2A_1B_1 \cdot \sin B_1A_1B_2 \cdot \sin A_2B_2B_1}{\sin B_2B_1C_1 \cdot \sin B_1C_2B_2 \cdot \sin C_2C_1A_1 \cdot \sin A_2A_1B_1 \cdot \sin C_1A_2C_2 \cdot \sin B_1A_1B_2 \cdot \sin A_2B_2B_1}
\]

and this is unity, since \( \angle C_1B_1C_2 = \angle C_1A_2C_2 \), and so on. Hence $AX$, $BY$, $CZ$ are concurrent.
Miscellaneous Exercises

1. Any straight line is drawn cutting two fixed intersecting straight lines, and the two angles on the same side of it are bisected. Find the locus of the point of intersection of the bisecting lines.

2. Find the locus of a point the difference of whose distances from two given intersecting straight lines is equal to a given length.

3. $P$ is a point inside a parallelogram $ABCD$ such that the area of the quadrilateral $PBCD$ is twice that of the figure $PBAD$. Show that the locus of $P$ is a straight line.

4. Two given straight lines meeting in $O$ are cut in $P$ and $Q$ by a variable third line. If the sum of $OP$, $OQ$ be constant, show that the locus of the middle point of $PQ$ is a straight line. [On $OP$, $OQ$ take $OA = OB = \frac{1}{2}(OP + OQ)$. $AB$ is the required locus.]

5. Two sides $AB$, $AD$ of a quadrilateral are given in magnitude and position and the area of the quadrilateral is given. Find the locus of the middle point of $AC$.

6. Given the base and the sum of the sides of a triangle. Find the locus of the feet of the perpendiculars let fall on the external bisector of the vertex angle from the ends of the base.

7. $BAC$ is a fixed angle of a triangle and (a) the sum; (b) the difference of the sides $AB$, $AC$ is given. Show that in either case the locus of the middle point of $BC$ is a straight line.

8. A straight line $XY$ moves parallel to itself so that the sum of the squares on the straight lines joining $X$ and $Y$ to a fixed point $P$ is constant. Find the locus of the middle point of $XY$.

9. Show that the locus of a point such that the sum of the squares on its distances from three given points may be constant is a circle the center of which lies at the centroid of the triangle formed by the three given points.

10. Prove that the locus of a point such that the sum of the squares on its distances from the vertices of a quadrilateral is constant is a circle the center of which coincides with the intersection of the lines joining the middle points of opposite sides of the quadrilateral.

11. Through one of the points of intersection of two fixed circles with centers $A$ and $B$, a chord is drawn meeting the first circle in $P$ and the other in $Q$. Find the locus of the point of intersection of $PA$ and $QB$. (The locus is an arc of a circle subtending the constant angle at the intersection of $PA$, $QB$.)

12. $ABC$ is a triangle inscribed in a circle, $P$ any point in the circumference of the circle, and $Q$ is a point in $PC$ such that the angle $QBC$ is equal to the angle $PBA$. Show that the locus of $Q$ is a circle.
13. **ABC** is a straight line, and any circle is described through $A$, $B$ and meeting in $P$, $P'$ the straight line bisecting $AB$ at right angles. Find the locus of the points in which $CP$ and $CP'$ cut the circle.

14. Two circles intersect in $A$, $B$. Through $A$ a chord $PAQ$ is drawn to meet both circles in $P$, $Q$. Find the locus of the center of the circle inscribed in the triangle $PBQ$.

15. Find the locus of the center of the circle which bisects the circumferences of two circles given in position and magnitude. (The locus is the perpendicular on the line of centers of the two circles.)

16. $A$ is a fixed right angle. Two equal distances $AB$, $AC$ are taken on the sides of angle $A$. If $C$ is joined to a fixed point $D$ and $BE$ is drawn perpendicular to $CD$, find the locus of $E$ for all positions of $B$, $C$.

17. Two circles intersect in $A$ and $B$ and a variable point $P$ on one circle is joined to $A$ and $B$. $PA$, $PB$, produced if necessary, meet the second circle in $Q$ and $R$. Prove that the locus of the center of the circle $PQR$ is a circle.

18. **AB** is a fixed chord of a given circle. $P$ is any point on the circumference. Perpendiculars $AC$ and $BD$ are drawn to $BP$ and $AP$ respectively. Find the locus of the middle point of $CD$.

19. Two circles touch and through the point of contact, a variable chord $AB$ is drawn cutting the circles in $A$, $B$. Show that the locus of the middle point of $AB$ is a circle. (Use the solution given in Problem 5.4.)

20. **AB** is a fixed chord of a circle and $X$ any point on the circumference. Find the locus of the intersection of the other tangents from $A$ and $B$ to the circle drawn, with center $X$, to touch $AB$.

21. Two equal circles of given radius touch, each, one of two straight lines which intersect at right angles and also touch each other. Find the locus of their point of contact with each other.

22. Given a fixed straight line $XY$ and a fixed point $A$. If $B$ is a moving point on $XY$ and $C$ is taken on $AB$ such that the rectangle $AC \cdot CB$ is constant, find the locus of $C$.

23. From a fixed point $P$, a straight line is drawn to meet the circumference of a fixed circle in $Q$. If $PQ$ is divided in $R$ such that $PR : PQ$ is constant, find the locus of $R$.

24. From a point $P$ inside a triangle $ABC$ perpendiculars $PD$, $PE$, $PF$ are drawn to $BC$, $CA$, $AB$ respectively. If the angle $EDF$ is equal to $A$, prove that the locus of $P$ is an arc of the circle passing through $B$, $C$ and the center of the circle circumscribing $ABC$.

25. If the two circles do not intersect orthogonally in Problem 5.4, find the locus of $E$ the middle point of $CD$ when (a) the circles are not equal; (b) the circles are equal.

26. $D$ is the middle point of a fixed arc $AB$ of a given circle. From $D$ any chord $DE$ is drawn in the circle cutting $AB$ in $F$. Show that the locus
of the center of the circle $AEF$ is a straight line. (Join $AD$, $DB$. The locus is the line joining $A$ to the center of circle $AEF$.)

27. Find the locus of the centers of the circles which pass through a given point and cut a given circle orthogonally.

28. If on each segment of a line, and on the same side of it, two equilateral triangles be described and their vertices joined to the opposite extremities of the line, the locus of the intersections of these lines is the circle circumscribing the equilateral triangle described on the other side of the line.

29. Through a fixed point which is equidistant from two parallel straight lines, a straight line is drawn terminated by the two fixed straight lines and on it as base is described an equilateral triangle. Show that the vertex of this triangle will lie on one of two straight lines.

30. $A$, $B$, $C$ are three points not in the same line. $D$ is any point on $BC$ such that if $AD$ is produced to $E$ then $AD : AE = BD : BC$. Find the locus of $E$.

31. On the external bisector of the angle $A$ of a triangle $ABC$, two points $D$, $E$ are taken such that $AD : AE = AB : AC$. Prove that the locus of the point of intersection of $DB$ and $EC$ is an arc of a circle drawn on $BC$ to subtend an angle $= \frac{1}{2}$ angle $A$.

32. $ABC$ is a triangle and $D$, $E$ are two points on $AB$, $AC$ respectively such that $BD : BA + CE : CA = BC^2$. Show that the locus of the point of intersection $P$ of $BE$, $CD$ is a circle. (Take point $F$ in $BC$ so that $AB : AC$ and prove that $ADPE$ is a cyclic quadrilateral, $P$ being the required point for locus, which is a circle on $BC$ subtending angle $BPC$.)

33. Given three points $A$, $B$, $C$ on one straight line and $D$ a moving point such that the angles $ADB$, $BDC$ are always equal, find the locus of $D$.

34. $AOB$ is a fixed diameter in circle with center $O$, and $C$ is a moving point along the circumference. If $D$ is taken on $BC$, such that the ratio of $BD : DC$ is constant, show that the locus of the point of intersection of $OD$, $AC$ is a circle.

35. The middle points of all chords of a circle which subtend a right angle at a fixed point lie on a circle.

36. $ABCD$ is a square. From $A$, $B$ two lines are drawn to meet $CD$ or produced in $E$, $F$ such that $CF : DE = CD^2$ is always constant. What is the locus of the point of intersection of $AE$, $BF$?

37. Two circles intersect in $A$, $B$. Through $A$ a line is drawn to meet the two circles in $P$, $Q$. If $PQ$ is divided by $R$ in a constant ratio, show that the locus of $R$ is a circle.

38. The base $BC$ and the vertex angle of the triangle $ABC$ are given. From $O$ the middle point of the base, $OA$ is drawn and produced to $P$ so that the ratio of $OP : OA$ is constant. Find the locus of $P$. 
39. Find the locus of a point inside the vertex angle $A$ of an isosceles triangle $ABC$, the distance of which from the base $BC$ is a mean proportional between its distances from the equal sides. (This is the circle drawn on $BC$ as a chord and touching $AB$, $AC$ at $B$, $C$ respectively.)

40. $OAB$ is a straight line rotating about its fixed end $O$. $A$, $B$ are two fixed points on the line and $D$ is another fixed point outside it. If the parallelogram $DACB$ is completed, construct the locus of $C$.

41. $P$ is a point in a segment of a circle described on a base $AB$, and the angle $APB$ is bisected by $PQ$ of such length that $PQ$ is equal to half the sum of $PA$ and $PB$. Prove that the locus of $Q$ for different positions of $P$ is a circle. (Let $PQ$ meet the circle in $E$. $E$ the mid-point of arc $AB$ is a fixed point and $AE = EB$. Since $PE : PA + PB = AE : AB$, then $EQ : PQ = AE - AD : AD$, a construction ratio. Hence the locus of $Q$ is a circle with center $E$.)

42. A triangle $ABC$ is inscribed in a fixed circle, the vertex $A$ being fixed and the side $BC$ given in magnitude. If $G$ be the centroid of the triangle $ABC$, show that the locus of $G$ is a circle.

43. A square is described with one side always on a given line, and one corner always on another. Find the locus of the corner which lies on neither.

44. One end $O$ of a straight line is fixed and the other $P$ moves along a given straight line. If $PQ$ be drawn at right angles to $OP$ such that $PQ/OP$ is constant, show that $Q$ traces out a straight line. (Draw $OA \perp$ the line on which $P$ moves. From $AP$ cut off $AB$ a fourth proportional to $OP$, $PQ$, $OA$. Then the locus of $Q$ is a line from $B \perp OB$.)

45. $ABC$ is a triangle inscribed in a circle and its vertex $A$ is fixed. A point $D$ is taken on $BC$ such that the ratio $AD^2 : DB \cdot DC$ is constant. Show that the locus of $D$ is a circle that touches circle $ABC$ at $A$.

46. $M$ is the center of a circle in which $AB$ is a fixed diameter. Any point $P$ is taken on $AB$ produced and $PQ$ is drawn tangent to the circle. If $MD$ is drawn perpendicular to the bisector of the angle $BPQ$, show that when $P$ moves along $AB$, $D$ traces a straight line parallel to $AB$. (Join $MQ$ and produce $MD$ to meet $PQ$ in $R$. Drop $RL \perp AB$.)

47. Two circles intersect in $A$, $B$ and $D$, $E$ are two points taken on their circumferences, such that the angle $DAE$ is constant. If $F$ divides $DE$ in the fixed ratio of $DF : FE$, find the locus of $F$.

48. Two circles intersect in $A$ and any straight line through $A$ meets them again in $P$, $Q$. Show that the locus of a point which divides $PQ$ in a constant ratio is a circle through the common points of the two circles.

49. Straight lines are drawn through the points $B$, $C$ of a triangle $ABC$, making with $AB$ and $AC$ produced angles equal to those made by $BC$ with $AB$ and $AC$. These lines meet in $A'$. Prove that $AA'$ passes through the center of the circle $ABC$ and that as $A$ moves round the circle, the locus of the orthocenter of $A'BC$ is a circle.
50. A is a fixed point inside a circle with center O. Any chord $BAC$ is drawn through $A$ and a semi-circle is constructed on $BC$ as a diameter. If $AD$ is drawn perpendicular to $BC$ meeting the semi-circle in $D$, show that the locus of $D$ is a circle which cuts circle $O$ in $H, G$ such that $HG$ always passes through the fixed point $A$.

51. Find the locus of a point such that the triangles formed by joining it to the ends of two given straight lines are equal to each other.

52. Find the locus of the foot of the perpendicular from a fixed point on a chord of a given circle which subtends a right angle at the fixed point. (Let $D$ be the fixed point and $AB$ be the chord of the given circle with center $O$. Draw $DE \perp AB$ and join $OD$. Bisect $OD$ in $F$ and it can be proved that $FE$ is constant and $F$ is fixed. Hence the locus is a circle with center $F$.)

53. Two circles are described one of which passes through a fixed point $A$ and has its center on a fixed line $AB$, and the other passes through a fixed point $C$ and has its center on a fixed line $CD$ parallel to $AB$. If the two circles touch, find the locus of their point of contact.

54. $AB$ is a fixed chord of a circle subtending a right angle at the center. $PQ$ is a variable diameter. Prove that the locus of the intersection of $AP$ and $BQ$ is a circle equal to the given circle.

55. A given straight line $AB$ is trisected in $C, D$. Lines $CP, DP$ are drawn through $C, D$ inclined at a given angle. $CP, DP$ are produced to $E, F$ so that $CP, DP$ are double of $PE, PF$ respectively. Find the locus of the intersection of $AF, BE$.

56. From a fixed point $D$ in the base $BC$ of a given triangle $ABC$, any line $DEF$ is drawn cutting $AB, AC$ or produced in $E, F$ respectively. The circles around $DEB, DFC$ intersect in $P$. Find the locus of $P$.

57. A point moves such that the sum of the squares on its distances from the sides of a square is equal to twice the square. Show that the locus of this point is a circle the radius of which bears a constant ratio to the side of the square. What is the condition for the radius of this locus circle to be equal to the side of the square?

58. $AOB$ is a fixed diameter of a circle with center $O$. $AC$ is a variable chord and $AC$ is divided at $D$ so that the ratio of $AD : DC$ is constant. Find the locus of the intersection of $OD, BC$.

59. The squares on the two sides of a triangle on a given base are together equal to five times the area of the triangle. Prove that the locus of the vertex of the triangle is a circle.

60. From a given external point $A$ a secant $AMN$ is drawn to a given circle. Find the locus of the intersection of the circles which pass through $A$ and touch the given circle in $M, N$, one externally, the other internally.

61. $ABC$ is a right-angled triangle at $A$. On the sides $AB, AC$ two squares $ABFG, ACHK$ are described. If $FG, HK$ be produced to meet in $P$, show that, as the right-angle changes its position, the hypotenuse $BC$ being
fixed, the locus of \( P \) will be a circle. (Draw \( BQ, CR \perp BC \) meeting \( FG \) and \( HK \) or produced in \( Q, R \). It can be proved that \( BQ = CR = BC \). Hence \( Q, R \) are fixed points. Therefore, the locus of \( P \) is a semi-circle on diameter \( QR \).)

62. In a triangle \( ABC \), the base \( BC \) and the length of the line \( CD \) which divides the side \( AB \) in a fixed ratio at \( D \) are given. Find the locus of the vertex \( A \) of the triangle.

63. \( A, B, C, D \) are four points on a straight line. Find the locus of a point \( P \) outside this line which will subtend equal angles with \( AB, CD \).

64. Find the locus of a point the difference of the squares on the distances of which from two opposite angular points of a square is equal to twice the rectangle contained by its distances from the other two angular points. (The locus is the circle circumscribing the square.)

65. \( A, B \) are the centers of two given circles of different diameters and \( M \) is a point which moves such that, if the two tangents \( MC, MD \) are drawn to circles \( A, B \) respectively, then \( 3MC^2 + MD^2 \) is always constant. Prove that the locus of \( M \) is a circle the center of which lies on \( AB \).

66. Show that in Problem 2.16 the line joining the middle points of the diagonals of a complete quadrilateral is the locus of the points which make, with every two opposite sides of the quadrilateral, two triangles (a) the sum of which equals half the area of the quadrilateral for the portion of this line inside the quadrilateral; (b) the difference of which equals half the area of the quadrilateral for that portion of the line outside the quadrilateral.

67. From \( B, C \) the angular points of a triangle right-angled at \( A \) are drawn straight lines \( BF, CE \) respectively parallel to \( AC, AB \) and proportional to \( AB, AC \). Find the locus of the intersection of \( BE \) and \( CF \).

68. Given the inscribed and circumscribed circles of a triangle, prove that the centers of the escribed circles in every position of the triangle lie on a circle. (Let \( ABC \) be one position of the \( \triangle, O_1, O_2, O_3 \) the centers of the escribed circles, \( I, O \) the centers of the inscribed and circumscribed circles. \( I \) is the orthocenter of \( \triangle O_1 O_2 O_3 \) and \( \odot ABC \) is its nine-point circle. If \( IO \) is produced to \( S \) so that \( OS = OI, S \) is the center of \( \odot O_1 O_2 O_3 \) and \( I, O \) are fixed points. Hence \( S \) is fixed, and \( SO_1 = 2 AO = \) constant.)

69. \( ABC \) is a triangle inscribed in a circle. Show that the tangents to the circle at \( A, B, C \) meet the opposite sides produced in three collinear points.

70. The non-parallel sides of a trapezoid are produced to meet in \( P \). Prove that the line joining \( P \) to the intersection of the diagonals of the trapezoid bisects the parallel sides.

71. (a) Prove that the lines joining the vertices of a triangle to the points of contact of the inscribed circle are concurrent; (b) show that the lines joining the vertices to the points of contact of the three escribed circles are also concurrent.
72. \( AD, BE, CF \) are any three concurrent lines drawn inside the triangle \( ABC \) to meet the opposite sides in \( D, E, F \). \( DE, EF, FD \) or produced meet \( AB, BC, CA \) in \( G, H, J \). Show that \( GHJ \) is a straight line.

73. \( P, Q \) are the centers of two circles intersecting in \( A, B \). Through \( A \) two perpendicular straight lines \( DAE, CAF \) are drawn meeting circle \( P \) in \( C, D \) and circle \( Q \) in \( E, F \) and cutting \( PQ \) in \( G, H \). Show that \( GD : GE = CH : HF \).

74. \( O \) is any point inside the triangle \( ABC \). If the bisectors of the angles \( BOC, COA, AOB \) meet the sides \( BC, CA, AB \) in \( D, E, F \) respectively, prove that \( AD, BE, CF \) are concurrent.

75. \( ABCD \) is a trapezoid in which \( AB, CD \) are the parallel sides. \( AD, BC \) when produced meet in \( E \), and \( AC, BD \) intersect in \( F \). From \( E \) a straight line \( GEH \) is drawn parallel to \( AB \) meeting \( AC, BD \) produced in \( G, H \). Show that \( AH, BG, EF \) are concurrent.

76. \( ABCD, EBFG \) are two parallelograms having the same angle \( B \). Show that if \( E \) lies on \( AB \), then \( AF, CE, GD \) are concurrent or parallel.

77. \( ABCD \) is a quadrilateral. If \( AB, CD \) meet in \( E \) and \( AD, BC \) in \( F \) and if \( ED' \) cut \( AD, BC \) in \( D', B' \) and \( FA'C' \) cut \( AB, CD \) in \( A', C' \) respectively, prove that \( AA' \cdot BB' \cdot CC' \cdot DD' = A'B \cdot B'C \cdot C'D \cdot D'A \).

78. \( ABCD \) is a square and \( EF, GH \) are two lines drawn parallel to \( AB, BC \), meeting \( BC, AD \) in \( E, F \) and \( AB, CD \) in \( G, H \). Show that \( BF, DG, CM \) are concurrent, \( M \) being the point of intersection of \( EF, GH \).

79. On the sides of a triangle \( ABC \), equilateral triangles \( BCD, CAE, ABF \) are constructed outside the triangle \( ABC \). Prove that \( AD, BE, CF \) are concurrent.

80. The escribed circle opposite to \( B \) touches \( BC \) at \( X \) and \( CA \) at \( Y \); also the inscribed circle touches \( AB \) at \( Z \). If \( X, Y, Z \) are collinear, show that \( A \) is a right angle.

81. \( AP, BQ, CR \) are three concurrent straight lines drawn from the vertices of a triangle \( ABC \) to the opposite sides. If the circle circumscribing the points \( P, Q, R \) cuts the sides \( BC, CA, AB \) again in \( X, Y, Z \), show that \( AX, BY, CZ \) meet in one point.

82. \( MNL \) is a right-angled triangle at \( M \). A square \( MNPQ \) is described outside the triangle, and \( LP \) cuts the perpendicular \( MT \) to \( NL \) in \( R \). Show that \( 1/\text{MR} = 1/\text{MT} + 1/\text{NL} \).

83. \( AD, BE, CF \) are the altitudes of the triangle \( ABC \) and \( P, Q, R, X, Y, Z \) are the middle points of \( BC, CA, AB \). \( AD, BE, CF \). Show that \( PX, QY, RZ \) are concurrent.

84. The interior angles \( A, B \) and the exterior angle \( C \) of a triangle \( ABC \) are bisected by lines which meet the opposite sides in \( D, E, F \). Prove that \( D, E, F \) are collinear.

85. If the ends of three unequal parallel straight lines be joined, two and two, toward the same parts, the joining lines intersect, two and two, in three collinear points.
86. Through a given point $P$ in the side $AB$ of a triangle $ABC$, draw a straight line meeting $BC$, $AC$ in $Q$, $R$ respectively so that $AR$ may be equal to $BQ$.

87. The altitudes of a triangle meet the sides $BC$, $CA$, $AB$ in $D$, $E$, $F$, $G$ is any point taken on $AD$ and $EG$, $FG$ produced meet $FD$, $ED$ in $P$, $Q$ respectively. Show that $EF$, $PQ$, $BC$ if produced will meet in one point.

88. Show that the middle points of the three diagonals of a complete quadrilateral are collinear.

89. $C$ is a moving point on the circumference of a circle with center $O$. $AB$ is a fixed diameter. If $AC$ is produced to $D$ such that $AC = CD$ and $OD$ is joined cutting $BC$ in $F$, find the locus of $F$. (See Problem 5.9.)

90. $D$, $E$, $F$ are the middle points of the sides $BC$, $CA$, $AB$ of a triangle $ABC$. On the base $BC$, two points $M$, $N$ are taken so that $BM = CN$. If $DE$, $AN$ intersect in $X$ and $ME$, $NF$ in $Y$, show that $X$, $Y$, $B$ are collinear.

91. Straight lines drawn from the vertices of a triangle $ABC$ parallel respectively to the sides of another triangle $A'B'C'$ in the same plane meet in a point $O$. Prove that the straight lines drawn through the vertices of the triangle $A'B'C'$ parallel respectively to the sides of the triangle $ABC$ also meet in a point $O'$, and that the three triangles $OBC$, $OCA$, $OAB$ are to each other in the same ratios as the triangles $O'B'C'$, $O'C'A'$, $O'A'B'$.

92. From any triangle $ABC$ another triangle $A'B'C'$ is formed by straight lines drawn through $A$, $B$, $C$ parallel to the opposite sides such that $A'$, $B'$, $C'$ are opposite to $A$, $B$, $C$ respectively. If any straight line through $A'$ meets $AB$, $AC$ in $F$, $E$ respectively, prove that (a) $B'F$, $C'E$ intersect in a point $D$ on $BC$; (b) the area of the triangle $DEF$ is a mean proportional between the areas of $ABC$, $A'B'C'$; (c) $AD$, $BE$, $CF$ are concurrent; (d) $A'D$, $B'E$, $C'F$ are parallel.

93. $ABC$ is a triangle inscribed in a circle. A transversal cuts the sides $BC$, $CA$, $AB$ externally in $D$, $E$, $F$. If the lengths of the tangents to the circle from $D$, $E$, $F$ be $x$, $y$, $z$, prove that $x \cdot y \cdot z = DC \cdot EA \cdot FB$.

94. $O$ is a point inside the triangle $ABC$. $OD$, $OE$, $OF$ are drawn each perpendicular to $OA$, $OB$, $OC$ respectively to meet $BC$, $CA$, $AB$ or produced in $D$, $E$, $F$. Show that $DEF$ is a straight line. (Produce $AO$, $BO$, $CO$ to meet the opposite sides in $G$, $H$, $J$. Use Menelaus’s Th. 5.13.)

95. The sides $BC$, $CA$, $AB$ of a triangle $ABC$ are bisected in $D$, $E$, $F$. A transversal $PQR$ cuts the sides $DE$, $DF$, $EF$ of the triangle $DEF$ and $AR$, $BQ$, $CP$ produced meet $BC$, $AC$, $AB$ respectively in $X$, $Y$, $Z$. Show that $X$, $Y$, $Z$ are collinear.

96. A line meets $BC$, $CA$, $AB$ of a triangle $ABC$ at $X$, $Y$, $Z$ and $O$ is any point. Show that $\sin BOX \cdot \sin COY \cdot \sin AOZ = \sin COX \cdot \sin AOV \cdot \sin BOZ$. 
97. If on the four sides $AB$, $BC$, $CD$, $DA$ of a quadrilateral $ABCD$ there be taken four points $L$, $M$, $N$, $R$ such that $AL \cdot BM \cdot CN \cdot DR = LB \cdot MC \cdot ND \cdot RA$, show that $LM$, $NR$ meet on $AC$ and $LR$, $MN$ on $BD$. (Apply Menelaus's theorem to $ABC$ and $LM$ also to $CDA$ and $NR$. Multiply and divide by the given relation.)

98. $ABC$, $A'B'C'$ are two triangles and $O$ is a point in their plane. Show that if $OA$, $OB$, $OC$ meet $B'C'$, $C'A'$, $A'B'$ in collinear points, then $OA'$, $OB'$, $OC'$ meet $BC$, $CA$, $AB$ in collinear points. (Apply the result obtained in Exercise 96.)

99. The escribed circles touch the sides $BC$, $CA$, $AB$ to which they are escribed at $X$, $Y$, $Z$. If $YZ$, $BC$ meet at $P$, $ZX$, $CA$ at $Q$, and $XY$, $AB$ at $R$, show that $P$, $Q$, $R$ are collinear. ($AZ \cdot BX \cdot CQ = ZB \cdot XC \cdot QA$, or $(s - b)(s - c) \cdot CQ = (s - a)(s - b) \cdot QA$. Then $CQ/QA = (s - a)/(s - c)$; $s$, $a$, etc., being the semi-perimeter and sides, and so on. Now multiply.)

100. $L$, $M$, $N$ are the centers of the circles escribed to the sides $BC$, $CA$, $AB$, of a triangle. Prove that the perpendiculars from $L$, $M$, $N$ to $BC$, $CA$, $AB$ respectively are concurrent.
CHAPTER 6

GEOMETRY OF LINES AND RAYS

HARMONIC RANGES AND PENCILS

Definitions and Propositions

**Definition.** If the rays PA, PB, PC, PD cut a line in A, B, C, D respectively, then the ratio \((AC/BC)/(AD/DB) = AC \cdot DB/BC \cdot AD\) is called the cross ratio of these four points, and is denoted by \((AB, CD)\). If this cross ratio = 1, then AC is divided harmonically in B, D.

**Definition.** If a straight line AB is divided harmonically in C, D; C, D are called harmonic conjugates with respect to A, B. Similarly, A, B are harmonic conjugates with respect to C, D, as explained in the next proposition. ACBD is called a harmonic range. AB is called the harmonic mean between AC, AD and DC the harmonic mean between AD, BD.

**Proposition 6.1.** If a straight line AB is divided harmonically in C, D, then CD is also divided harmonically in B and A (Fig. 162).

\[
\text{Figure 162}
\]

Since \(AD : BD = AC : BC\) (hypothesis), \(\therefore BD : AD = BC : AC\)

and \(DB : BC = AD : AC\). Therefore, DC is divided harmonically in B and A.

**Proposition 6.2.** If C, D be harmonic conjugates with respect to A, B and AB be bisected in 0, then \(OB^2\) is equal to \(OC \cdot OD\), and the converse.

1. Since \(AC : CB = AD : DB\), \(\therefore (AC - CB) : (AC + CB) = (AD - BD) : (AD + BD)\) or \(OC : OB = OB : OD\). Hence \(OB^2 = OC \cdot OD\).

2. Let AB be bisected in 0 and let \(OB^2 = OC \cdot OD\), AB will be divided harmonically in C, D. \(\therefore OB^2 = OC \cdot OD\), \(\therefore OC : OB = OB : OD\) and \((OB + OC) : (OB - OC) = (OD + OB) : (OD - OB)\). Therefore, \(AC : BC = AD : BD\).

**Proposition 6.3.** The geometric mean between two straight lines is a mean proportional between their arithmetic and harmonic means.

Let \(AD, BD\) be the straight lines. Bisect \(AB\) in \(O\) and with center \(O\) and
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radius $OA$ describe a semi-circle $APB$. Draw tangent $DP$, and $PC \perp AB$ (Fig. 163). Then $2\, OD = AD + BD$. \therefore OD is the arithmetic mean between $AD$, $BD$. Also $AD \cdot DB = DP^2$. \therefore $AD : DP = DP : BD$. \therefore $DP$ is the geometric mean between $AD$, $BD$. Again in the right-angled $\triangle OPD$, $OD \cdot OC = OP^2 = OB^2$. Therefore, $ACBD$ is a harmonic range (Prop. 6.2) and $DC$ is the harmonic mean between $AD$, $BD$. And in the right-angled $\triangle OPD$, $OD : PD = PD : DC$.

**COROLLARY:** If from a point $D$ in a produced diameter $AB$ of a circle, a tangent $DP$ be drawn to the circle and $PC$ be drawn perpendicular to $AB$, $(ACBD)$ is a harmonic range.

**Definition.** Any number of straight lines passing through a point $P$ are said to form a pencil; the point $P$ is called the vertex of the pencil and each of the lines is called a ray of the pencil.

Any straight line cutting the rays of a pencil is called a transversal.

In a pencil of four rays, if each ray passes through a point of a harmonic range, it is called a harmonic pencil. The rays which pass through conjugate points of the range are called conjugate rays. $(P \cdot ABCD)$ denotes a pencil whose vertex is $P$ and whose rays pass through the points $A$, $B$, $C$, $D$.

**Proposition 6.4.** Any straight line drawn parallel to one of the rays of a harmonic pencil is divided into two equal parts by the other three rays, and conversely.

1. Let $EFG$ be drawn $\parallel PD$ one of the rays of harmonic pencil $(P \cdot ACBD)$. Through $C$ draw $MCN \parallel EG$ or $PD$ (Fig. 164). \therefore $\triangle BCN$, $PBD$ are similar, \therefore $CN : PD = CB : BD$. Similarly, $CM : PD = AC : AD$. But $AC : BC = AD : BD$ (hypothesis). \therefore $AC : AD = BC : BD$.

2. Let $EFG$ drawn $\parallel PD$, one of the rays of the pencil $(P \cdot ACBD)$ of four rays, be divided into two equal parts by the other three rays $PA$, $PC$, $PB$. $(P \cdot ABCD)$ will be a harmonic pencil. Draw any transversal $ACBD$; through $C$ draw $MCN \parallel EG$ or $PD$. \therefore $EF = FG$. \therefore $MC = CN$. \therefore $MC : PD = NC : PD$. Also, $MC : PD = AC : AD$ and $NC : PD = CB : BD$. \therefore $AC : AD = BC : BD$ or $AC : BC = AD : BD$.

3. Likewise, if $EG$ be divided into two equal parts by three rays $PA$, $PC$, 

\[ \text{Figure 163} \]
PB of the harmonic pencil \((P \cdot ACBD)\), then it is easy to prove that \(EG\) will be \(\parallel PD\).

**Corollary 1.** Any transversal is cut harmonically by the rays of a harmonic pencil.

![Figure 164](image)

**Corollary 2.** If one of the outside rays \(DP\) be produced through \(P\) to \(D'\), then \((P \cdot BCAD')\) is a harmonic pencil and by producing the other rays in succession four other harmonic pencils may be said to be formed.

**Corollary 3.** If three points of a harmonic range are given, the fourth can be found.

(i) Let the required point be an outside one. Given \(A, C, B,\) find \(D\). From any point \(P\) draw \(PA, PC, PB\). Draw \(EFG\) so as to be bisected in \(F\) and \(PD \parallel EG\). \(D\) is the required point.

(ii) Let the required point be an inside one. Given \(A, C, D,\) find \(B\). From any point \(P\) draw \(PA, PC, PD\). From any point \(E\) in \(PA\) draw \(EF \parallel PD\) and produce \(EF\) to \(G\) so that \(FG = EF\). Join \(PG\) and produce it to meet \(AD\) in \(B\). \(B\) is the required point.

**Proposition 6.5.** If one ray of a harmonic pencil bisects the angle between the other pair of rays, the ray conjugate to the first ray is at right angles to it, and conversely.

1. Let \((P \cdot ACBD)\) be a harmonic pencil and let \(PC\) bisect \(\angle APB\). Through \(C\) draw \(ECF \perp PC\) (Fig. 165). \(\therefore EC = CF\). \(\therefore EF\) is \(\parallel PD\). \(\therefore PD\) is \(\perp PC\).

2. Conversely, if the conjugate rays \(PC, PD\) of the harmonic pencil \((P \cdot ACBD)\) be \(\perp\) to each other, then it is easy to show that they will bisect the \(\angle s\) between the other two rays \(PA, PB\).

3. The internal and external bisectors of an angle form with the arms of the angle a harmonic pencil.

**Proposition 6.6.** Any diagonal of a complete quadrilateral is divided harmonically by the other two diagonals.
Let $EF$ be the third diagonal of the complete quadrilateral $ABCDEF$ and let it be divided by the other diagonals $AC$, $BD$ in $G$, $H$. Through $C$ draw $(KCLMN \parallel AB$ (Fig. 166). In $\triangle FHB$, $AHB$, $KL : LN = FB : AB$


In the same way, by drawing through $O$ a $\parallel$ to $AD$ meeting $AB$ in $Q$, it may be proved that the pencil $(E \cdot AQB F)$ is harmonic. Hence $AC$ is divided harmonically in $O$, $G$.

**Proposition 6.7.** If four rays $OA$, $OB$, $OC$, $OD$ of a pencil $(O \cdot ABCD)$ are cut by two transversals $ABCD$, $EFGH$ respectively, then $(BC, AD) = (FG, EH)$ and this is true for any other transversals (Fig. 167).

\[
(BC, AD) = \frac{BA}{BD} \frac{DC}{AC} = \frac{\triangle AOB}{\triangle BOD} \frac{\triangle COD}{\triangle AOC}
\]
\[
\frac{AO \cdot OB \sin AOB}{OD \cdot OB \sin BOD} = \frac{OD \cdot OC \sin COD}{AO \cdot OC \sin AOC} \\
= \frac{\sin AOB}{\sin BOD} \cdot \frac{\sin COD}{\sin AOC} \\
= \frac{EF}{GH} \cdot \frac{CH}{GE} \\
= \frac{(FG, EH)}{(F'G', E'H')}. 
\]

Since this cross ratio depends only on the pencil \((O \cdot ABCD)\) and not on any particular transversal, we may denote the common cross ratio as \(O(AC, BD)\).

**Solved Problems**

6.1. D, E are two points on the straight line AX and F any point on another line AY. FD, FE cut a third line AZ in B, C. If M is any point on AZ and DM, EM meet AY in P, Q, show that PC, QB meet on AX.

**Construction:** Produce DC, EB to meet AY in R, S. Let PC, QB meet AX in N, N' (Fig. 168).

**Proof:**

\[
(AN, DE) = C(AN, DE) \\
= (AP, RF) \quad \text{(from Prop. 6.7)} \\
= D(AP, RF) = (AM, CB) \\
= E(AM, CB) = (AQ, FS) = B(AQ, FS) \\
= (AN', DE). 
\]
Hence $N'$ should coincide with $N$. Therefore, $PC$, $QB$ meet on $AX$. This is one of Pascal's theorems.

6.2. In a triangle $ABC$, three concurrent lines $AP$, $BQ$, $CR$ meet in $S$ and the opposite sides in $P$, $Q$, $R$ respectively. $PU$ meets $QR$ in $X$, $QU$ meets $RP$ in $Y$, $RU$ meets $PQ$ in $Z$. Show that $AX$, $BY$, $CZ$ are concurrent.

Construction: Produce $AX$, $BY$, $CZ$ to meet $BC$, $CA$, $AB$ in $E$, $F$, $G$ respectively (Fig. 169).

Proof: Let $AP$, $BQ$, $CR$ meet $QR$, $PR$, $PQ$ in $H$, $K$, $L$ respectively. $AB$, $AE$, $AP$, $AC$ are rays cut by the two transversals $RQ$, $BC$. .

6.3. The straight lines joining the excenters of a triangle to the middle points of the opposite sides are concurrent.

CONSTRUCTION: Let \( O_1, O_2, O_3 \) be the three excenters of \( \triangle ABC \) opposite to the vertices \( A, B, C \) respectively and \( A', B', C' \) be the mid-points of \( BC, CA, AB \). Join the excenters and produce the lines \( O_1A', O_2B', O_3C' \) to meet \( O_2O_3, O_1O_3, O_1O_2 \) in \( R, L, N \) (Fig. 170).

\[ \text{Proof:} \quad O_1O_2 \text{ passes through } C, \quad O_2O_3 \text{ through } A, \quad \text{and } O_3O_1 \text{ through } B. \quad O_1A', O_2B', O_3B' \text{ are also the altitudes of } \triangle O_1O_2O_3; \quad \text{hence } \triangle ABC \text{ is the pedal of } \triangle O_1O_2O_3. \text{ According to Prop. 6.7, } (A'D, BC) = (RA, O_3O_2) \text{ and } (B'E, AC) = (LB, O_3O_1). \text{ Also } (C'F, AB) = (NC, O_2O_1). \text{ But, since } A', B', C' \text{ are the mid-points of the sides, } DC/BD = (AO_2/AO_3)(RO_3/RO_2) \text{ and } AE/CE = (O_3B/BO_1)(LO_1/LO_3) \text{ and } BF/AF = (O_1C/CO_2)(O_2N/O_1N). \text{ Since } (DC/BD)(AE/CE)(BF/AF) = 1 \text{ and } (A'O_2/AO_3)(O_3B/BO_1)(O_1C/CO_2) = 1, \therefore (RO_3/RO_2)(LO_1/LO_3)(O_2N/O_1N) = 1. \therefore O_1A', O_2B', O_3C' \text{ are concurrent.} \]
6.4. ABC is a triangle and AD, BE, CF are the altitudes. If \( O_1, O_2, O_3 \) are the excenters opposite to A, B, C respectively, show that \( O_1D, O_2E, O_3F \) are concurrent.

**Construction:** Join \( O_1O_2, O_2O_3, O_3O_1 \) passing through C, A, B respectively. Let \( AO_1, BO_2, CO_3 \) and \( O_1D, O_2E, O_3F \) cut BC, CA, AB, \( O_2O_3, O_3O_1, O_1O_2 \) in G, H, J, L, M, N respectively (Fig. 171).

![Figure 171](image)

**Proof:** ABC is the pedal \( \Delta \) of \( \triangle O_1O_2O_3 \). \((GD, BC) = (AL, O_3O_2) \) and \((EH, AC) = (MB, O_3O_1) \); also, \((FJ, AB) = (NC, O_2O_1) \). Multiplying the ratios, we get

\[
\frac{DC}{BG} \cdot \frac{EA}{CH} \cdot \frac{FB}{AJ} = \frac{LO_2}{BD} \cdot \frac{O_3A}{GC} \cdot \frac{MO_3}{EC} \cdot \frac{BO_1}{AH} \cdot \frac{O_1N}{FA} \cdot \frac{O_2C}{JB}
\]

But

\[
\frac{DC}{BD} = 1, \quad \frac{EA}{EC} = 1, \quad \frac{FB}{FA} = 1, \quad \frac{O_3A}{AO_2} \cdot \frac{BO_1}{BO_3} = 1, \quad \frac{O_2C}{CO_1} = 1, \quad \frac{BG}{GC} \cdot \frac{CH}{AH} \cdot \frac{AJ}{JB} = 1.
\]

Hence \( \frac{LO_2}{LO_3} \cdot \frac{MO_3}{MO_1} \cdot (O_1N/O_2N) = 1 \). Therefore, \( O_1D, O_2E, O_3F \) are concurrent.
6.5. From a point $M$ inside or outside a triangle $ABC$, a transversal is drawn to cut the sides $BC$, $CA$, $AB$ in $D$, $E$, $F$ respectively. If $MA$, $MB$, $MC$, $EF$, $DF$, $ED$ are bisected in $A'$, $B'$, $C'$, $D'$, $E'$, $F'$ respectively, show that $A'D'$, $B'E'$, $C'F'$ meet in one point.

**Construction:** Let $DEF$ cut the sides of $\triangle A'B'C'$ in $K$, $T$, $S$. Produce $F'C'$, $C'M$ to meet $A'B'$ in $R$, $Y$, $A'D'$, $A'M$ to meet $B'C'$ in $L$, $X$ and $B'E'$, $B'M$ to meet $A'C'$ in $N$, $Z$ (Fig. 172).

![Figure 172](image)

**Proof:** Since $A'$, $B'$, $C'$ are the mid-points of $AM$, $BM$, $CM$, $\therefore S$, $T$, $K$ are the mid-points of $MF$, $ME$, $MD$. But $(MD', ST) = (XL, B'C')$ or $(MT/MS) (D'S/D'T) = (XC'/B'X)(LB'/LC')$. $\therefore D'T = \frac{1}{2} MF = MS$, $\therefore MT = D'S$. $\therefore MT^2/MS^2 = (XC'/B'X)(LB'/LC')$. Similarly, $MK^2/MT^2 = (B'Y/A'Y)(A'R/RB')$ and $MS^2/MK^2 = (A'Z/ZC')(C'N/NA')$. But since $(XC'/B'X)(B'Y/A'Y)(A'Z/ZC') = 1$ (Th. 5.11), $\therefore$ multiplying yields $(LB'/LC')(A'R/RB')(C'N/NA') = 1$. Hence $A'D'$, $B'E'$, $C'F'$ are concurrent.

**ISOGONAL AND SYMMEDIAN LINES—BROCARD POINTS**

**Definitions and Propositions**

**Definition.** Two straight lines $AP$, $AP'$ are said to be isogonal or conjugate lines of the straight lines $AB$, $AC$ if the angles $PAP'$ and $BAC$ have the same bisectors.

**Proposition 6.8.** If $P$ and $P'$ are any points on the isogonal lines $AP$, $AP'$ of the lines $AB$, $AC$ and if perpendiculars $PN$, $P'N'$ are drawn to $AB$ and perpendiculars $PM$, $P'M'$ to $AC$, then $PN \cdot P'N' = PM \cdot P'M'$ and, conversely, if $PN \cdot P'N' = PM \cdot P'M'$, then $AP$, $AP'$ are isogonal lines of $AB$, $AC$ (Fig. 173).
Since the angles $PAB$, $P'AC$ are equal, the $\triangle s$ $PAN$, $P'AM'$ are similar. Hence $\frac{AP}{PN} = \frac{AP'}{P'M'}$ or $\frac{AP}{AP'} = \frac{PN}{P'M'}$. So, from $\angle PAC = \angle P'AB$, $\frac{AP}{AP'} = \frac{PM}{P'N'}$. Hence $PN \cdot P'M' = PM \cdot P'N'$; i.e., $PN \cdot P'N' = PM \cdot P'M'$. Conversely, if $PN \cdot P'N' = PM \cdot P'M'$, $AP$ and $AP'$ are isogonal lines. For, if not, let a parallel to $AC$ through $P'$ cut the isogonal line of $AP$ at $P''$. Then, by the first part, $PN \cdot P''N'' = PM \cdot P''M'' = PM \cdot P'M'$ (since $P''M'' = P'M'$) = $PN \cdot P'N'$ (by hypothesis). Hence $P'N' = P''N''$ or $P'P''$ is also parallel to $AB$. Hence $P''$ coincides with $P'$. Hence $AP$ and $AP'$ are isogonal lines.

Proposition 6.9. If $AP$, $BP$, $CP$ meet in a point $P$, the isogonal lines of $AP$ with respect to $AB$, $AC$, of $BP$ with respect to $BC$, $BA$ and of $CP$ with respect to $CA$, $CB$ meet in a point $P'$ such that the product of the perpendiculars from $P$ and $P'$ on each of the sides is the same (Fig. 174).
Let the isogonals of $AP$ and $BP$ meet at $P'$. Draw the $s PL, P'L'$ on $BC$, and $PM, P'M'$ on $CA$ and $PN, P'N'$ on $AB$. It is easily shown that $CP'$ is the isogonal of $CP$. Since $AP, AP'$ are isogonal, $PN \cdot P'N' = PM \cdot P'M'$. So from $BP, BP'$, $PN \cdot P'N' = PL \cdot P'L'$. Hence $PM \cdot P'M' = PL \cdot P'L'$. Therefore, $CP'$ is the isogonal of $CP$. Hence the isogonals of $AP, BP, CP$ meet in a point $P'$ such that $PL \cdot P'L' = PM \cdot P'M' = PN \cdot P'N'$.

Such points $P, P'$ are said to be isogonal or conjugate points with respect to the triangle $ABC$.

**Definition.** The isogonal point of the centroid $G$ of a triangle $ABC$ is called the symmedian or Lemoine point and is denoted by $K$. The lines $AK, BK, CK$ isogonal to $AG, BG, CG$ are called the symmedians.

**Proposition 6.10.** Show that the perpendiculars from $K$ on the sides of $ABC$ are proportional to the sides.

Let the $s$ from $G$ on the sides $BC, CA, AB$ denote by $a, b, c$ be $a', b', c'$. Then since $\triangle BGC = \triangle CGA = \triangle AGB$, $\therefore aa' = bb' = cc'$. But according to Proposition 6.9, if $s$ from $K$ on $a, b, c$ be $p, q, r$, then $a'p = b'q = c'r$. Hence $p : q : r = a : b : c$.

Obvious cases of isogonal points with respect to a triangle are the circumcenter and orthocenter. Also, there are four points, and only four, each of which is isogonal to itself with respect to $ABC$ and these are the incenter and the three excenters.

**Proposition 6.11.** One and only one point $\Omega$ can be found such that $\angle BA\Omega = \angle AC\Omega = \angle CB\Omega$. Also one and only one point $\Omega'$ can be found such that $\angle AB\Omega' = \angle BC\Omega' = \angle CA\Omega'$, where $\Omega, \Omega'$ are called the Brocard points and are isogonal with respect to $ABC$. $\angle BA\Omega = \angle AB\Omega'$ is called the Brocard angle and is denoted by $\omega$.

If a point $\Omega$ exists such that $\angle BA\Omega = \angle AC\Omega = \angle CB\Omega$, then the
circle \( A\Omega C \) must touch \( BA \) at \( A \), since \( \angle BA \Omega = \angle AC \Omega \). So the circle \( C \Omega B \) must touch \( AC \) at \( C \) (Fig. 175). Hence \( \Omega \) is determined as the second intersection of the circles (1) through \( C \) and touching \( BA \) at \( A \) and (2) through \( B \) and touching \( AC \) at \( C \). Hence two points \( \Omega \) cannot exist.

Again the second intersection \( \Omega \) of the above circles is such that \( \angle BA \Omega = \angle AC \Omega \) (from the first circle) = \( \angle CB \Omega \) (from the second circle). Hence one such point exists. Similarly for \( \Omega' \).

Also \( \Omega \) and \( \Omega' \) are isogonal points. For, if not, let \( \Omega'' \) be isogonal to \( \Omega \). Then \( \angle BA \Omega = \angle CA \Omega'' \), \( \angle AC \Omega = \angle BC \Omega'' \), \( \angle CB \Omega = \angle AB \Omega'' \). Hence, since \( \angle BA \Omega = \angle AC \Omega = \angle CB \Omega \), then \( \angle CA \Omega'' = \angle BC \Omega'' = \angle AB \Omega'' \). Hence \( \Omega'' \) has the property of \( \Omega' \) and therefore coincides with it, since there is only one point \( \Omega' \).

**Solved Problems**

6.6. Three lines are drawn through the vertices of a triangle meeting the opposite sides in collinear points (Fig. 176). Show that their isogonal conjugates with respect to the sides also meet the opposite sides in collinear points.

![Figure 176](image)

**Proof:** Let \( D, E, F \) be the collinear points and \( AD', BE', CF' \) the isogonal conjugates. Then \( \sin ACF \cdot \sin BAD \cdot \sin CBE = \sin FCB \cdot \sin DAC \cdot \sin EBA \). But \( \angle ACF = \angle F'CB \), and so on. Hence \( \sin F'CB \cdot \sin D'AC \cdot \sin E'BA = \sin ACF' \cdot \sin BAD' \cdot \sin CBE' \). Therefore, \( D', E', F' \) are collinear (see Problem 5.30).

6.7. The sum of the squares of the perpendiculars from a point to the sides of a triangle has its least value when the point is the symmedian point; evaluate its magnitude.
Proof: Let the sides $BC$, $CA$, $AB$ be $a$, $b$, $c$ and the $l$s from the symmedian point $K$ be $p$, $q$, $r$. Then $pa + qb + rc$ is constant, being twice the area of the triangle. But $(p^2 + q^2 + r^2)(a^2 + b^2 + c^2) = (pa + qb + rc)^2 + (pb - qa)^2 + (qc - rb)^2 + (ra - pc)^2$. Hence $(p^2 + q^2 + r^2)$ is minimum when $pb = qa$, $qc = rb$, $ra = pc$, i.e., when $p/a = q/b = r/c$ or $p : q : r = a : b : c$, i.e., at the symmedian point $K$ (Prop. 6.10). In this case, the minimum value of the sum $(p^2 + q^2 + r^2) = (pa + qb + rc)^2/(a^2 + b^2 + c^2) = 4(\triangle ABC)^2/(a^2 + b^2 + c^2)$.

6.8. If the symmedians $AP$, $BQ$, $CR$ of a triangle $ABC$ meet the opposite sides $BC$, $CA$, $AB$ in $P$, $Q$, $R$, show by applying Ceva’s theorem that they are concurrent at the symmedian or Lemoine point $K$ (Fig. 177).

![Figure 177](image)

Proof: According to Prop. 6.10, if $p$, $q$, $r$ be the $l$s from $K$ on $BC$, $CA$, $AB$, : : : $p : q : r = a : b : c$ or $p/a = q/b = r/c = k$. $BP/PC = \triangle BKA/\triangle CKA = r \cdot c / q \cdot b = k \cdot c / k \cdot b = c^2 / b^2$ and similarly $CQ/QA = a^2 / c^2$ and $AR/RB = b^2 / a^2$. Hence multiplying yields $(BP/PC)(CQ/QA)(AR/RB) = 1$. Therefore, $AP$, $BQ$, $CR$ meet in one point which is the symmedian or Lemoine point $K$ (Ceva’s Th. 5.11).

6.9. The external sides of squares described outwardly on the sides of the triangle $ABC$ meet at $A'$, $B'$, $C'$. Show that $AA'$, $BB'$, $CC'$ are the symmedians of $ABC$.

**Construction:** Let $AA'$, $BB'$ meet in $K$. Draw $KL$, $KM$, $KN \perp s B'C', C'A', A'B'$ (Fig. 178).

**Proof:** From Prop. 6.10, if $K$ is the symmedian point of $\triangle ABC$, : : : $p : q : r = a : b : c$, where $a$, $b$, $c$ denote $BC$, $CA$, $AB$. Hence $(p + a) : (q + b) : (r + c) = a : b : c$. Now in $\triangle A'B'C'$, $KL : KN = BE : BH = a : c$. : : : $(p + a) : (r + c) = a : c$. 


Hence $BB'$ is the locus of points the distances of which from $BC$, $AB$ are in the ratio $a : c$. Similarly, $AA'$, $CC'$ are the loci of the points the distances of which from $AC$, $AB$ and $BC$, $CA$ respectively have the ratios $b : c$ and $a : b$.

$$(p + a) : (q + b) : (r + c) = a : b : c.$$

Hence $AA'$, $BB'$, $CC'$ meet at the symmedian point $K$.

6.10. In Prop. 6.11, show that if $S_2$, $S_2'$ are the Brocard points of the triangle $ABC$, then $A\Omega \cdot B\Omega \cdot C\Omega = A\Omega' \cdot B\Omega' \cdot C\Omega'$.

Proof: In Fig. 175, $\angle A\Omega C = [180 - \omega - (A - \omega)] = 180^\circ - A$. Now in \(\triangle A\Omega C\), $A\Omega / \sin \omega = AC / \sin A\Omega C$ or $A\Omega = AC \sin \omega / \sin A$, and so on; and $A\Omega' = AB \sin \omega / \sin A$, and so on. Hence $A\Omega \cdot B\Omega \cdot C\Omega = AC \cdot AB \cdot BC \cdot \sin^3 \omega / \sin A \cdot \sin B \cdot \sin C = A\Omega' \cdot B\Omega' \cdot C\Omega'$.

Miscellaneous Exercises

1. $(BC, XX')$, $(CA, YY')$, $(AB, ZZ')$ are harmonic ranges. Show that if $AX$, $BY$, $CZ$ are concurrent, then $X'$, $Y'$, $Z'$ are collinear.

2. $AS$, $BS$, $CS$ meet $BC$, $CA$, $AB$ at $P$, $Q$, $R$. $QR$, $RP$, $PQ$ meet $BC$, $CA$, $AB$ at $X$, $Y$, $Z$. Show that $(BC, PX)$, $(CA, QY)$, $(AB, RZ)$ are harmonic ranges and that $X$, $Y$, $Z$ are collinear, also that $AX$, $CZ$, $BQ$ are concurrent. (From the quadrilateral $ARSA$, $(BC, PX)$ is harmonic. Hence $BP : PC = BX : CX$. Now use Exercise 1.
3. If \((AB, CD)\) is a harmonic range and \(P\) any point collinear with \(AB\), show that \(2 \frac{PB}{AB} = \frac{PC}{AC} + \frac{PD}{AD}\).

4. If in Exercise 3, \(U\) and \(V\) bisect \(AB\) and \(CD\), then \(PA \cdot PB + PC \cdot PD = 2 \frac{PU \cdot PV}{AB}\).

5. If \((AB, CD)\) and \((AB', C'D')\) are harmonic ranges, show that \(BB', CD'\) and \(C'D\) meet in a point.

6. \(TA, TB\) are drawn touching a circle in \(A, B\). From any point \(C\) in \(AB\) produced, the straight line \(CDEF\) is drawn touching the circle in \(E\) and cutting \(TA, TB\) in \(D, F\). Prove that \(CE\) is cut harmonically in \(D\) and \(F\).

7. \(ABC\) is a triangle inscribed in a circle. The tangent at \(A\) meets \(BC\) produced in \(D\). Prove that the second tangent from \(D\) is cut harmonically by the sides \(AB, AC\) and its point of contact with the circle. [Let \(DE\) be the second tangent. Then \((A \cdot DCEB)\) is a harmonic pencil (Prop. 6.6), etc.]

8. \(ABCD\) is a complete quadrilateral whose diagonals intersect in \(O\). \(BA, CD\) meet in \(E\) and \(BC, AD\) meet in \(F\). If \(AC, BD\) intersect \(EF\) in \(O', O''\), show that each two of \(O, O', O''\) are harmonic points with respect to two corners of \(ABCD\) and that if a parallel is drawn through \(O\) to \(O'O''\) to meet \(AB, DC, AD, BC\) in \(G, H, I, J\), then \(OG = OH\) and \(OI = OJ\), and similarly with \(O'\) and \(O''\).

9. The three triangles \(ABC, A'B'C', A''B''C''\) are such that \((A'A'', BC), (B'B'', CA),\) and \((C'C'', AB)\) are harmonic. Show that \(BC, B'C', B''C''\) are concurrent and so are \(CA, C'A', C''A''\) and \(AB, A'B', A''B''\).

10. Through a given point \(O\) draw a line cutting the sides \(BC, CA, AB\) of a triangle in points \(A', B', C'\) such that \((OA', B'C')\) is harmonic.

11. \(A, B, C, D\) are four collinear points. Find the locus of \(P\) when the circles \(ABP\) and \(CDP\) touch at \(P\).

12. With any point on a given circle as center a second circle is described cutting the first. Prove that any diameter of the second circle is divided harmonically by the first circle and the common chord or chord produced.

13. The base \(BC\) of a triangle \(ABC\) is bisected in \(D\) and a point \(O\) is taken in \(AD\) or \(AD\) produced. Show how to draw a straight line through \(D\) terminated by the sides of the triangle such that the segments into which it is divided at \(D\) may subtend equal angles at \(O\). (Draw \(OG \perp AD, AG \parallel BC\). Let \(GD\) meet \(AC, AB\) in \(E, F\). \(EDF\) is the required line. Use Prop. 6.4.2.)

14. Prove that if a straight line \(PQRS\) be drawn intersecting the sides of a square in \(P, Q, R, S\) so that \(PQ \cdot RS = PS \cdot QR\), \(PQRS\) will touch the inscribed circle of the square.

15. Find the locus of a point from which the tangents to two given circles have a given ratio to each other.
16. Prove that the circles on the diagonals of a complete quadrilateral as diameters cut orthogonally the circle circumscribing the triangle formed by the diagonals. (Use Prop. 6.6.)

17. In Prop. 6.9, show that in the figure the points $L, L', M, M', N, N'$ are concyclic.

18. If $AB$ is a diameter of a circle and $P$ a point on the circumference, find the position of $P$ when $l \cdot PA + m \cdot PB$ is greatest, $l$ and $m$ being given.

19. If $AP, AQ$ are two isogonals of $AB, AC$ which cut $BC$ at $P, Q$ show that $AP^2 : AQ^2 = BP \cdot PC : BQ \cdot QC$.

20. $ABC$ is a triangle and $AP, AQ$ are isogonal conjugate lines with respect to $AB$ and $AC$, $P$ and $Q$ being on the circle $ABC$. Show that the isogonal conjugate point of $P$ is the point at infinity on $AQ$. (Let $P'$ be the point at infinity on $AQ$. It is sufficient to prove that $\angle ABP' = \angle CBP$. Now $BP' || AQ$. Hence $ABP' = BAQ = CAP = CBP$.)

21. If $P$ and $P'$ are isogonal points of $ABC$, show that $AP' : BP' : CP' = AP \cdot PL : BP \cdot PM : CP \cdot PN$, where $PL, PM, PN$ are the perpendiculars on the sides $BC, CA, AB$.

22. In a triangle right-angled at $A$, prove that $K$ bisects the altitude $AD$, $K$ being the symmedian point.

23. Given the side $BC$ and the angle $CBA$, show that the locus of one Brocard point is a circle. (In Fig. 175, suppose $BC$ is given and also the $\angle BCA$. Then the locus of $\Omega$ is the circle touching $CA$ at $C$ and passing through $B$. So in the given case, the locus of $\Omega'$ is the circle touching $BA$ at $B$ and passing through $C$.)

24. If a parallel to $BC$ through $A$ cuts at $R$ the circle which passes through $C$ and touches $BA$ at $A$, prove that $BR$ cuts this circle again in $\Omega$.

25. $A\Omega, B\Omega, C\Omega$ meet the circle $ABC$ again at $P, Q, R$. Show that the triangles $ABC, PQR$ are congruent and have a common Brocard point.

26. If $PQ$ is the tangent from $P$ to a given circle and if any other line through $P$ meets the circle in $R$ and $R'$, show that the triangles $PQR, PQR'$ have the same Brocard angle and have one Brocard point in common.
CHAPTER 7

GEOMETRY OF THE CIRCLE

SIMSON LINE

Definitions and Propositions

Proposition 7.1. The projections of any point which lies on the circumcircle of a triangle on the sides of the triangle are collinear and conversely, if the projections of a point on to the sides of a triangle are collinear, this point lies on the circumcircle of the triangle. This line is called the pedal or Simson line of the point with respect to the triangle and is equidistant from the given point and the orthocenter of the triangle. Let \( L, M, N \) be the projections of \( P \) on \( BC, CA, AB \). Let also \( AK, G \) be an altitude and the orthocenter of \( \triangle ABC \). Produce \( AK \) to meet the \( \bigcirc \) in \( H \). Join \( PH \) cutting \( BC, LN \) in \( F, J \). Join \( FG, PA, PC \). Let also \( PG \) cut \( LN \) in \( Q \) (Fig. 179). Now, suppose that \( P \) lies on the circumcircle of \( ABC \). Then, \( L, M, N \) will be collinear, if \( PML + PMN = 2 \) right angles, which is true, since \( \angle PMN = \angle PAN \) (PNAM is cyclic), \( \angle PAN = \angle PCL \) (PABC is cyclic) and \( \angle PML + \angle PCL = 2 \) right angles (PMLC is cyclic). Conversely, if \( LMN \) is a straight line, it is simple to show that \( PABC \) is cyclic. Again, \( \angle PLJ = \angle PCA = \angle PHA \)
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= ∠JPL (since AH, PL are ⊥s BC). ∴ JL = JP = JF. Since it can be proved that GK = KH and GK is ⊥BC, ∴ ∠GFK = ∠HFK. But ∠HFK = ∠JFL = ∠JLF. ∴ ∠GFK = ∠JLF. ∴ FG is || LMN. ∴ Q is the mid-point of PG. Hence the ⊥s from P, G on LMN are equal.

Solved Problems

7.1. Find a point such that the feet of the four perpendiculars from it to the sides of a given quadrilateral may be collinear.

Analysis: Let ABCD be a quadrilateral. Produce BA, CD to meet in E and BC, AD in F. Now, the two △s ECB, FAB combine the four sides of the quadrilateral ABCD and both △s have the sides AB, BC in common (Fig. 180). Hence the circumcircles on △s ECB, FAB intersect in B, P such that the Simson lines of P with respect to △s ECB, FAB, which are LNQ, LMQ, coincide, since both have two points in common.

Synthesis: It is obvious that if the two circumcircles ECB, FAB intersect in P, then P will be the required point.

7.2. If p and r are the pedal lines of P and R with respect to the same triangle, then the angle between p and r is equal to the angle subtended by PR at any point on the circumcircle of the triangle. For in Fig. 180, ∠MLB = right angle − ∠PLM = right angle − ∠PCA. Hence ∠pr = ∠MLB − ∠M'LB = ∠RCA − ∠PCA = ∠RCP.

7.3. If the perpendiculars PL, PM, PN from any point P on the circumcircle intersect in B, P such that the Simson lines of P with respect to △s ECB, FAB, which are LNQ, LMQ, coincide, since both have two points in common.

Analysis: Let ABCD be a quadrilateral. Produce BA, CD to meet in E and BC, AD in F. Now, the two △s ECB, FAB combine the four sides of the quadrilateral ABCD and both △s have the sides AB, BC in common (Fig. 180). Hence the circumcircles on △s ECB, FAB intersect in B, P such that the Simson lines of P with respect to △s ECB, FAB, which are LNQ, LMQ, coincide, since both have two points in common.

Synthesis: It is obvious that if the two circumcircles ECB, FAB intersect in P, then P will be the required point.

7.2. If p and r are the pedal lines of P and R with respect to the same triangle, then the angle between p and r is equal to the angle subtended by PR at any point on the circumcircle of the triangle. For in Fig. 180, ∠MLB = right angle − ∠PLM = right angle − ∠PCA. Hence ∠pr = ∠MLB − ∠M'LB = ∠RCA − ∠PCA = ∠RCP.

7.3. If the perpendiculars PL, PM, PN from any point P on the circumcircle intersect in B, P such that the Simson lines of P with respect to △s ECB, FAB, which are LNQ, LMQ, coincide, since both have two points in common.

Analysis: Let ABCD be a quadrilateral. Produce BA, CD to meet in E and BC, AD in F. Now, the two △s ECB, FAB combine the four sides of the quadrilateral ABCD and both △s have the sides AB, BC in common (Fig. 180). Hence the circumcircles on △s ECB, FAB intersect in B, P such that the Simson lines of P with respect to △s ECB, FAB, which are LNQ, LMQ, coincide, since both have two points in common.

Synthesis: It is obvious that if the two circumcircles ECB, FAB intersect in P, then P will be the required point.

7.2. If p and r are the pedal lines of P and R with respect to the same triangle, then the angle between p and r is equal to the angle subtended by PR at any point on the circumcircle of the triangle. For in Fig. 180, ∠MLB = right angle − ∠PLM = right angle − ∠PCA. Hence ∠pr = ∠MLB − ∠M'LB = ∠RCA − ∠PCA = ∠RCP.

7.3. If the perpendiculars PL, PM, PN from any point P on the circumcircle intersect in B, P such that the Simson lines of P with respect to △s ECB, FAB, which are LNQ, LMQ, coincide, since both have two points in common.

Analysis: Let ABCD be a quadrilateral. Produce BA, CD to meet in E and BC, AD in F. Now, the two △s ECB, FAB combine the four sides of the quadrilateral ABCD and both △s have the sides AB, BC in common (Fig. 180). Hence the circumcircles on △s ECB, FAB intersect in B, P such that the Simson lines of P with respect to △s ECB, FAB, which are LNQ, LMQ, coincide, since both have two points in common.

Synthesis: It is obvious that if the two circumcircles ECB, FAB intersect in P, then P will be the required point.

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of a triangle $ABC$ to the sides $BC$, $CA$, $AB$ are produced to meet the circle in $E$, $F$, $G$, show that $AE$, $BF$, $CG$ are parallel to the pedal line $LMN$. Similarly, if $P'$, another point on the circle, lies on the perpendicular from $B$ to $AC$, then the tangent at $B$, $AE'$, $CG'$ will be parallel to the pedal line of $P'$ with respect to $ABC$, where $E'$, $G'$ are the intersections of the perpendiculars $P'L'$, $P'N'$ to $BC$, $BA$ with the circle.

**Construction:** Join $PA$, $PC$. $L'M'N'$ is the pedal line of $P'$ with respect to $ABC$ (Fig. 181).

![Figure 181](image)

**Proof:** $\therefore$ $PMAN$ is cyclic, $\therefore$ $\angle NPA = \angle NMA = \angle GCA$. $\therefore$ $CG$ is $\parallel MN$. Also, $\angle MPA = \angle MNA = \angle FBA$. $\therefore$ $BF$ is $\parallel MN$. Again, since $PMLC$ is cyclic, $\therefore$ $\angle PLM = \angle PCM = \angle PEA$. $\therefore$ $AE$ is $\parallel LMN$. Hence $AE$, $BF$, $CG$ are $\parallel LMN$. Similarly, $AE'$ is $\parallel L'M'$. But $\angle L'CM' = \angle L'P'M' = \angle E'P'B = \angle E'AB$. $\therefore$ $\angle F'BA = \angle BCA$ or $\angle L'CM'$ (since $BF'$ is tangent to $\odot$ at $B$), $\therefore$ $\angle E'AB = \angle F'BA$. $\therefore$ $BF'$ is $\parallel L'M'$. Similar to $AE'$, $CG'$ is $\parallel L'M'N'$.

7.4. From a point $P$ on the circumcircle of the triangle $ABC$, perpendiculars $PL$, $PM$, $PN$ are drawn to $BC$, $CA$, $AB$. Find the position of $P$ in order that $ML = MN$.

**Analysis:** Assume that $LMN$ is the required pedal line of $P$ with respect to $\triangle ABC$. Join $PA$, $PC$ (Fig. 182). Now, $PCLM$ is cyclic. $\therefore$ $PM = PC \sin PCM = PC \sin PLM$ (since $\angle PMC$ is right angle). $\therefore$ $PM/\sin PLM = PC = ML/\sin MPL = ML/\sin C$ (law of sines for $\triangle PLM$). Hence $ML = PC \sin C$. Similarly, $MN = AP \sin A$. If $LM = MN$ as required, $\therefore$ $AP/PC = \sin C/\sin A = AB/BC$ (law of
sines for $\triangle ABC$ and can be proved also by dropping $BE \perp AC$). Therefore, $AP/PC = \text{construction ratio of } AB/BC$, and $P$ can be determined by the intersection of the Apollonius $\bigcirc$ which divides $AC$ internally and externally into the fixed ratio of $AB:BC$ at $D, F$, with $\bigcirc ABC$.

**SYNTHESIS:** Draw the Apollonius $\bigcirc$ on $DF$ as diameter cutting $\bigcirc ABC$ in $B, P$. \( \therefore \) $AP/PC = AD/DC = AB/BC = \sin C/\sin A$. \( \therefore \) $PC \sin C = AP \sin A$ or $ML = MN$. Likewise, if it is required that $ML = k \cdot MN$, where $k$ is a fixed integer, then draw the Apollonius $\bigcirc$ that divides $AC$ in the same way into the ratio $AB : k \cdot BC$.

7.5. The orthocenters of the four triangles formed by four lines are collinear.

**CONSTRUCTION:** Let the four given lines be $ABE, DCE$ meeting in $E$ and $BCF, ADF$ meeting in $F$. Let also $O_1, O_2, O_3, O_4$ be the orthocenters of the $\triangle$s $BCE, ABF, ADE, DCF$. Draw the two circumcircles $BCE, DCF$ to intersect in $P$. From $P$ draw $PL, PM, PN, PQ \perp \triangle ABE, DCE, BCF, ADF$ and join $O_1P, O_2P, O_3P, O_4P$ (Fig. 183).

**Proof:** Since $P$ is the point of intersection of the circumcircles $BCE, DCF$, \( \therefore \) $LMNQ$ is one straight line, which is the Simson line of $P$ with respect to both $\triangle$s $BCE$ and $DCF$ (according to Prop. 7.1). Again, it is easy to prove that $ADPE, ABPF$ are cyclic quadrilaterals. Hence in the same way as of Prop. 7.1, $LMNQ$ can be proved to be the Simson line of both $\triangle$s $ADE, ABF$. But, since $O_1P, O_2P, O_3P, O_4P$ are bisected by $LMNQ$ in the four $\triangle$s $BCE, ABF, ADE, DFC$
(Prop. 7.1) and $LMNQ$ is a straight line, \( \therefore O_1, O_2, O_3, O_4 \) are collinear.

**RADICAL AXIS—COAXAL CIRCLES**

**Definitions and Propositions**

**Proposition 7.2.** Find the locus of a point, the tangents from which to two given circles are equal. (i) If the circles intersect, it is clear that the extension of their common chord is the required locus. (ii) If they touch one another, the common tangent at their point of contact is the required locus. (iii) If they do not meet, let $C, C'$ be the centers of the $\odot$s and suppose the circle $C$ is the greater. Let $P$ be any point on the locus. Draw the tangents $PT, PT'$ and $PO \perp CC'$. Join $PC, PC', CT, C'T'$ (Fig. 184). Hence $OC^2 - OC'^2$
GEOMETRY OF THE CIRCLE

\[ CP^2 - C'P^2 = CT^2 - C'T'^2. \therefore PT = PT', \therefore \text{the } \perp \text{ PO divides } CC' \text{ into segments such that the difference of the squares on them = the difference of the squares on the corresponding radii.} \therefore O \text{ is a fixed point and } PO \text{ therefore a fixed straight line and hence the required locus.}

**Definition.** The line \( PO \) is called the radical axis of the two circles. From \( O \) draw the tangents \( AO, OA' \). These are equal. Take \( OL, OL' \) each equal to \( OA \) or \( OA' \) and join \( PL, PL' \). Hence \( PT^2 = PC^2 - CT^2 = PO^2 + OC^2 - CA^2 = PO^2 + OA^2 = PO^2 + OL^2 = PL^2. \therefore PT = PL. \)

Similarly, \( PT' = PL' \). \therefore PT = PL = PL' = PT'.

**Corollary:** The circle described with any point on the radical axis of two circles as center, and a tangent to one of the circles from this point as radius, will cut both circles orthogonally and if the circles do not meet will pass through two fixed points \( L, L' \) called the limiting points.

**Proposition 7.3.** If there be three circles whose centers are not in the same straight line, their radical axes taken two and two are concurrent. Let \( A, B, C \) be the centers of the \( \bigcirc s, r_1, r_2, r_3 \) their radii. Let the radical axes of \( A, B; B, C; C, A \) cut \( AB, BC, CA \) in \( E, F, G \) respectively (Fig. 185). If \( EM, EN \)

\[
\begin{align*}
&= r_1^2 - r_2^2 \text{ (Prop. 7.2)},
&= r_2^2 - r_3^2, 
&= r_3^2 - r_1^2. 
& \text{ Adding gives } (AE^2 - EB^2) + (BF^2 - FC^2) + (CG^2 - GA^2) = 0. 
& \text{ Therefore the } \perp \text{s from } E, F, G \text{ to } AB, BC, CA \text{ respectively are concurrent (Problem 2.20).}
\end{align*}
\]

**Corollary 1.** If three circles intersect, two and two, their common chords are concurrent and if three circles touch each other, two and two, the three common tangents at their points of contact are concurrent.
Definition. The point of concurrence of the radical axes of three given circles taken two and two is called the radical center of the circles.

Corollary 2. The circle described with the radical center of three circles as center and a tangent from that point to any one of them as radius cuts the three circles orthogonally.

Corollary 3. If a variable circle be described through two given points to cut a given circle, the common chord passes through a fixed point on the line which joins the given points.

Proposition 7.4. Find the radical axis of two given circles. (i) If the circles intersect, the common chord extended is their radical axis. (ii) If the circles touch, the common tangent at their point of contact is their radical axis. (iii) If the circles do not intersect, describe a circle cutting each of the given circles, with any point not in the line of the centers of the circles as center. Draw the common chords. These produced intersect in the radical center of the three circles. Therefore, the perpendicular drawn from this point to the line joining the centers of the given circles is their radical axis. Note the relationship of this proposition to Proposition 7.2.

Definition. If any number of circles, taken two and two, have the same radical axis, they are called coaxal. These coaxal circles reduce to the points $L, L'$ (Prop. 7.2) called the limiting points of the family of the circles.

Proposition 7.5. If one coaxial system of circles has its centers on the common chord of another system of intersecting circles, then the two common points of intersection of the second system are the limiting points of the first.

It can be shown that if a system of circles has its centers collinear and has also a common orthogonal circle, it is a coaxal system. Now in Fig. 186, one
system consists of \( L \) (a point circle) and the circles \( a_2, a_3, \ldots \), which increase in size until eventually the radical axis \( i \) is reached. On the other side we get \( L' \), \( a_5, a_6, \ldots \), also approaching the radical axis as a limit. The orthogonal system of \( b_1 \) on \( LL' \) as diameter (which is the smallest circle), \( b_2, b_3, \ldots \) and \( b_4, b_5, \ldots \), both series approaching the line \( LL' \) as a limit.

Solved Problems

7.6. If \( PN \) is the perpendicular from any point \( P \) to the radical axis of two circles the centers of which are \( A, B \), and \( PQ, PR \) are the tangents from \( P \) to the circles, then \( PQ^2 - PR^2 = 2 \, PN \cdot AB \).

Construction: Draw \( PM \perp AB \), and join \( PA, AQ, PB, BR \) (Fig. 187).

![Figure 187](image)

Proof: Let \( a, b \) be the radii of \( \odot A, B \).

\[
PQ^2 - PR^2 = (PA^2 - a^2) - (PB^2 - b^2)
= (PA^2 - PB^2) - (a^2 - b^2)
= AM^2 - MB^2 + OB^2 - AO^2 \quad \text{(since } OA^2 - a^2 = OB^2 - b^2)\]

\[
= (AM - MB)(AM + MB) + (OB - AO)(OB + AO)
= (AM - MB + OB - AO)AB
= (AM - AO + OB - MB)AB
= 2 \, OM \cdot AB = 2 \, PN \cdot AB.
\]

If \( P \) lies on one of the circles, for instance \( B \), then \( PR = 0 \) and \( PQ^2 = 2 \, PN \cdot AB \).

7.7. Given two circles, construct a system of circles coaxal with them.

Construction: (i) If the given \( \odot\)'s intersect, any \( \odot \) through their points of intersection will be coaxal with them.
(ii) If the circles touch, any circle touching them at their point of contact will be coaxal with them.

(iii) If the given circles $ABD$, $EFG$ do not meet, take $C$, $C'$ their centers, and draw their radical axis $OL$ (Prop. 7.4). From $O$ draw tangent $OA$. With center $O$ and radius $OA$ describe circle $AEH$; from any point $H$ on its circumference draw tangent $HK$ meeting $CC'$ in $K$ (Fig. 188). With center $K$ and radius $KH$ describe circle $HNM$. $OH$ touches circle $HNM$ and $OH = OA$. $\therefore$ $OL$ is radical axis of circles $ABD$, $HNM$. Hence circles $ABD$, $EFG$, $HNM$ are coaxal. Similarly, by drawing tangents from other points on the circumcircle of circle $AEH$, any number of circles can be described coaxal with $ABD$, $EFG$.

7.8. Construct a circle which will pass through two given points and touch a given circle.

Construction: Let $A$, $B$ be the given points and $CDE$ the given circle. Describe a circle passing through $A$, $B$ and cutting a given circle $C$, such that the three circles have a common point $F$. Denote the radii of these circles as $OA$, $OB$, and $OC$, respectively. Let $G$ be the center of the circle $CDE$. The intersection of the line $AG$ and the circle $ABC$ is the center $O$ of the required circle.
D. Let $AB, CD$ meet in $F$. $F$ is the radical center of $\odot s\ CDE, ABC$, and the required $\odot$. Draw tangent $FG$ to $\odot CDE$. $FG$ is the radical axis of $\odot CDE$ and the required $\odot$ (Fig. 189). Hence $\odot$ through $A, B, G$ is the required $\odot$. Describe this $\odot$. Since $FA \cdot FB = FC \cdot FD = FG^2$, $\therefore$ $FG$ touches $\odot ABG$. $\therefore$ $\odot ABG$ touches $\odot CDE$. If the other tangent from $F$ to $\odot CDE$ meet that $\odot$ in $G'$, the $\odot$ through $A, B, G'$ will also touch given $\odot CDE$.

7.9. $AD, BE, CF$ are perpendiculars from $A, B, C$ to the opposite sides of the triangle $ABC$. $EF, FD, DE$ meet $BC, CA, AB$ respectively in $L, M, N$. Show that $L, M, N$ are collinear and that the straight line through them is perpendicular to the line joining the orthocenter and the circumcenter.

**Construction:** Draw the nine-point circle of $\triangle ABC$ with center $S$ the mid-point of $OG$, $O, G$ being the circumcenter and orthocenter (Fig. 190).

![Figure 190](image)

**Proof:** Since the nine-point circle passes through $D, E, F$, then $\odot DEF$ is the nine-point circle of $\triangle ABC$. But $BDEA$ is cyclic (since $\angle ADB = \angle AEB = \text{right angle}$). Therefore, $N$ lies on the radical axis of these two $\odot s$. Similarly, $L, M$ lie also on the radical axis of these two $\odot s$. Hence $LMN$ is one straight line and is the radical axis of these two $\odot s$. $\therefore$ The line of the centers $OSG$ is $\perp$ the radical axis of the two $\odot s\ LMN$.

7.10. $ABCD$ is a quadrilateral in which the angles $B, D$ are equal. Two points $E, F$ are taken on $BC, CD$ respectively, such that $\triangle AED, \triangle AFB$ are equal. Show that the radical axis of the two circles described on $BF, DE$ as diameters passes through $A$. 
CONSTRUCTION: Draw $EP, FQ \parallel AD, AB$ respectively meeting $AB, AD$ produced in $P, Q$. Join $BD, PQ, DP, BQ$. Let $O$s on $BF, DE$ as diameters cut $AB, AD$ in $G, H$ respectively and join $EH, FG$ (Fig. 191).

**Figure 191**

Proof: $\therefore EP \parallel AD, \therefore \triangle AED = \triangle APD$. Also, $FQ \parallel AB$. $\therefore \triangle AFB = \triangle AQB$. Since $\triangle AED = \triangle AFB, \therefore \triangle APD = \triangle AQB$. Eliminating common $\triangle ADB$ yields $\triangle BPD = \triangle DQB$. $\therefore DB \parallel PQ$. $\therefore AB : BP = AD : DQ$ (1). But $\triangle$s $BEP, DFQ$ are similar, $\therefore BP : BE = DQ : DF$ (2). Multiplying (1) and (2) gives $AB : BE = AD : DF$. But $\angle B = \angle D$. $\therefore \triangle$s $ABE, ADF$ are similar. $\therefore \angle BAF = \angle DAE$. Since $\angle BGF = \angle DHE = \text{right angle}$, $\therefore \triangle$s $AGF, AHE$ are similar. $\therefore GF : AG = HE : AH$. $\therefore \triangle AED = \triangle AFB, \therefore AB : GF = AD : HE$. $\therefore AG : AB = AH : AD$,

Hence the tangent from $A$ to $\odot BFG = \text{tangent from } A \text{ to } \odot DEH$. Therefore, $A$ lies on the radical axis of these two $O$s, which, in this case, is their common chord $LK$.

**POLES AND POLARS**

Definitions and Propositions

Definition. If the radius $CA$ of a circle, $C$ being the center, be divided
internally and externally in P and D so that the rectangle $CP \cdot CD$ is equal to
the square on CA, the straight line drawn through either P or D perpendicular
to CA is called the polar of the other point with respect to the circle and that
other point is called the pole of the perpendicular. P and D are sometimes
called conjugate poles. Hence when the pole is inside the circle, the polar does
not meet the circle and when the pole is on the circumference of the circle its
polar is the tangent drawn at the pole.

**Proposition 7.6.** If a line cuts a circle, the point of intersection of the
tangents drawn at the points where the line cuts the circle is the pole of the
line.

Let $TT'$ cut $\bigcirc ATT'$ of center C. Draw tangents $TP$, $T'P$. Join $CT$,
$CP$ cutting $TT'$ in D (Fig. 192). Now, $\angle CTP$ is right and $TT' \perp CP$.

\[ \therefore CP \cdot CD = CT^2. \therefore P \text{ is the pole of } TT'. \text{ Hence when the pole is outside}
the circle, its polar cuts the circle and coincides with the chord of contact of
tangents to the circle drawn from the pole.

**Proposition 7.7.** If from every point in a given straight line pairs of
tangents be drawn to a given circle, all the chords of contact intersect in the
pole of the given line.

Let $MN$ be the given line, $P$ its pole with respect to the $\bigcirc$ with center C.
From any point $M$ in $MN$ draw tangents $MA$, $MB$ and draw $PE \perp CM$
(Fig. 193). $\because \angle s PDM, PEM$ are right, $\therefore DPEM$ is cyclic. $\therefore CE \cdot CM$
$= CP \cdot CD = \text{square on radius}$. $\therefore PE$ is the polar of $M$. Hence $AB$
passes through $P$. This proposition may also be enunciated thus: the
polars of all points in a straight line intersect in the pole of the line.

**Corollary 1.** If the polar of $P$ passes through $Q$, then the polar
of $Q$ passes through $P$. Such points as $P$, $Q$ are called conjugate points
with respect to circle.

**Corollary 2.** The polar of the intersection of two straight lines is
the line joining their poles.

**Corollary 3.** The poles of all straight lines meeting in a point
lie on the polar of that point.
Proposition 7.8. The locus of the intersection of pairs of tangents to a circle drawn at the extremities of a chord which passes through a given point is the polar of the point.

Let P (Fig. 193) be the given point, C the center of given circle. Through P draw any chord AB. Join CP and draw FPG perpendicular to CP. Draw also tangents AM, BM, FD, GD. Let CM, AB intersect in E. Join MD. \( \therefore CP \cdot CD = \text{square on radius} = CM \cdot CE, \therefore DPEM \text{ is cyclic and } \angle PEM \text{ is right.} \)

Hence MD is the polar of P, and this is the locus of M.

This proposition may also be enunciated thus: the locus of the poles of all secants to a given circle which pass through a given point is the polar of the point.

Corollary: The pole of the straight line joining two points is the point of intersection of the polars of the points.

Proposition 7.9. Any straight line drawn through a fixed point and cutting a given circle is divided harmonically by the point, the circle, and the polar of the point, and the converse.
(i) Let $PCEF$ cut $MN$ the polar of $P$ in $E$. Draw $PBDOA$ through the
center $O$. Join $OF, OM, OC, DF, DC, PM$; produce $FD$ to $G$ (Fig. 194).
\[ \angle PMO \text{ is right, } \therefore PO\cdot PD = PM^2 = PC\cdot PF. \therefore CDOF \text{ is cyclic. } \]
\[ \therefore \angle PDC = \angle OFC = \angle OCF = \angle ODF \text{ in some segment. } \therefore DM \text{ bisects } \angle FDC. \therefore DP \text{ bisects the adjacent } \angle CDG. \therefore PCEF \text{ is a harmonic range. A similar proof holds when } P \text{ lies inside the } \odot. \]

(ii) If the chord $CF$ be divided harmonically in $E$ and $P$, then the polar of $P$
passes through $E$. This is easily proved indirectly.

Solved Problems

7.11. The distances of two points from the center of a circle are proportional
to the distance of each point from the polar of the other.

CONSTRUCTION: Let $DN$ be the polar of $P$ and $EM$ the polar of $Q$
with respect to a circle whose center is $C$. Draw $PF \perp CQ$, $PM \perp EM$, $QG \perp PC$, and $QN \perp DN$ (Fig. 195).

![Figure 195](image)

Proof: Since $QGP$, $QFP$ are right angles, \[ \therefore QGFP \text{ is cyclic. } \therefore GC\cdot CP = QC\cdot CF. \] Also, \[ CD\cdot CP = \text{ square on radius } = CQ\cdot CE. \therefore \]
\[ GC\cdot CP + CD\cdot CP = QC\cdot CF + QC\cdot CE \text{ or } CP\cdot GD = QC\cdot EF. \] Hence
\[ PC : QG = EF : GD = PM : QN. \] This is called Salmon's theorem.

7.12. (i) Parallel tangents to a circle at $P$, $Q$ meet the tangent at the point $R$
in $S$, $T$. $PQ$ meets this tangent in $U$. Show that $RV$ is the polar of $U$, $V$ being
the intersection of $PT$ and $QS$ (Fig. 196).

Proof: Let a line $\parallel$ to $PS$ through $V$ cut $PQ$ at $M$. \[ \therefore VM \perp PQ. \] Again the harmonic points of the quadrangle $PSTQ$ are $U$, $V$ and
the point $I$ at infinity on $PS$. Hence $V(PQ, UI)$ is harmonic. \[ \therefore (PQ, 
UM) \text{ is harmonic. Hence the polar of } U \text{ passes through } M \text{ and it is}
\perp PQ. \therefore \text{ It is } VM \text{ which also passes through } R. \]
(ii) $M$, $N$ are the projections of a point $R$ on a circle on two perpendicular diameters. $G$ is the pole of $MN$ and $U$, $L$ are the projections of $G$ on the diameters. Show that $UL$ touches the circle at $R$.

Proof: Since the polar of $G$ passes through $M$, the polar of $M$ passes through $G$ and, being $\perp$ the radius $CM$, is $GU$, \:. passes through $U$. Hence the polar of $U$ passes through $M$ and therefore $RM$. So the polar of $L$ is $RN$. Hence the polar of $R$ (on $RM$ and $RN$) is $UL$. Hence $UL$ is the tangent at $R$.

7.13. $ABCD$ is a quadrilateral inscribed in a circle. $AD$, $BC$ produced meet in $E$ and $AB$, $DC$ meet in $F$. $O$ is the point of intersection of $AC$, $BD$. Show that each vertex of the triangle $OEF$ is the pole of the opposite side (Fig. 197).
Proof: Let $BD, FE$ produced meet in $H$ and let $FO, BE$ meet in $K$. 
$\therefore AC$ is cut harmonically in $O, G$ (Prop. 6.6), $\therefore$ the polar of $O$ passes through $G$ (Prop. 7.9) and $\therefore BD$ is cut harmonically in $O, H$, $\therefore$ the polar of $O$ passes through $H$. $\therefore EF$ is the polar of $O$. Again, $\therefore$ the polar of $O$ passes through $E$, $\therefore$ the polar of $E$ passes through $O$ (Prop. 7.7, Cor. 1). $\therefore F(BODH)$ is a harmonic pencil, $\therefore BC$ is cut harmonically in $K, E$. $\therefore$ The polar of $E$ passes through $K$. $\therefore FO$ is the polar of $E$ and similarly $EO$ is the polar of $F$.

7.14. If a quadrilateral be inscribed in a circle and tangents be drawn at its angular points forming a circumscribed quadrilateral, the interior diagonals of the two quadrilaterals are concurrent and their third diagonals coincident (Fig. 198).

![Figure 198](image)

Proof: Let the tangents at $A, C$ meet in $N$. $\therefore G$ is the pole of $AD$ and $K$ the pole of $BC$, $\therefore E$ is the pole of $GK$.

Similarly, $F$ is the pole of $HL$. $\therefore$ The intersection of $GK, HL$ is the pole of $EF$. But the intersection of $AC, BD$ is the pole of $EF$. $\therefore$ The four diagonals $AC, BD, GK, HL$ intersect in $O$. Again, $\therefore M$ is the pole of $BD$ and $N$ is the pole of $AC$. $\therefore MN$ is the pole of $O$. $\therefore EF, MN$ coincide. It can be deduced also that:

1. $E$ is the pole of $GK$, but $E$ is also the pole of $FO$. Therefore, $GK$ passes through $F$. Similarly, $HL$ passes through $E$.

2. $HL$ is the polar of $F$, therefore $H(AOBF)$ is a harmonic pencil. Hence $FMEN$ is a harmonic range; i.e., the extremities of the third diagonal of one quadrilateral are harmonic conjugates to the extremities of the other.
3. Also $O(AHBF)$ is a harmonic pencil; i.e., the diagonals of the two quadrilaterals form a harmonic pencil.

7.15. Construct a cyclic quadrilateral given the circumscribing circle and the three diagonals.

ANALYSIS: Let $ABCD$ be the required quadrilateral inscribed in $O$. Let the tangents to the $O$ at $A, B, C, D$ meet in $H, K, L, G$. $HK$, $GL$ produced meet in $M$ and $GH, LK$ produced meet in $N$. Also $AD$, $BC$ meet in $E$ and $AB, DC$ in $F$. \[ MN \] coincides with $EF$ (Problem 7.14). Join $IM, IN$ cutting $BD, AC$ at right angles in $Q, P$ (Fig. 199).

Hence $Q, P$ are the mid-points of $BD, AC$. Produce $PQ$ to meet $EF$ in $R$. Since the mid-points of the diagonals of a complete quadrilateral are collinear (Problem 2.16), $R$ is the mid-point of $EF$. $M, N$ are the poles of $BD, AC$. \[ IP \cdot IN = IQ \cdot IM = \text{square on radius of } O. \] $AC, BD$ are given, \[ \therefore \text{their distances } IP, IQ \text{ from the center } I \text{ are known. Therefore, } IN, IM \text{ are known. But } IR^2 = \text{tangent}^2 \text{ from } R \text{ to } I + \text{radius}^2 \text{ of } I, \] and since the tangent from $R$ to $O = \text{half the third diagonal } EF$ (Problem 3.23), $IR^2 = RE^2 + \text{radius}^2 \text{ of } I \text{ and hence is known.}$

But since the transversal $PQR$ cuts the sides of $\triangle IMN$, \[ (RM/RN)(IQ/QM)(NP/IP) = 1 \text{ (Menelaus' Th. 5.13). } \] \[ RM/RN = (QM/IQ)(IP/NP), \] and since $QM = IM - IQ = \text{known}$, so also $NP$ is a known quantity. Hence $RM : RN$ is a known ratio. This reduces the problem to drawing the $\triangle IMN$ given $IM, IN$ and the
length of the line $IR$ which divides $MN$ into a known ratio $RM : RN$. Draw $MS \parallel IN$. \[ MS : IN = RM : RN \] and is known and \[ IS : IR = MN : RN \] and is also known, and since $IM$ is known, the triangle $IMS$ can be constructed knowing the three sides.

**Synthesis:** Take the known length $IR$ on $IS$ produced and join $RM$ and produce it to meet the $\parallel$ to $MS$ from $I$ in $N$. On $IN$, $IM$, take $P$, $Q$ with the known lengths $IP$, $IQ$. Describe the given $\odot$ with center $I$ and from $P$, $Q$ draw $AC$, $BD \perp IN$, $IM$. $ABCD$ is the required quadrilateral.

**Similitude and Inversion**

**Definitions and Propositions**

**Definition.** If the straight line joining the centers of two circles be divided internally and externally in the points $S'$, $S$ in the ratio of the radii of the circles, $S'$ is called the internal center and $S$ the external center of similitude of the circles. The circle described on $SS'$ as diameter is called the circle of similitude.

It follows from the definition that if two circles touch each other, the point of contact is their internal center of similitude if the contact is external, their external center of similitude if the contact is internal (Fig. 200). If $O$, $O'$ are the centers of the circles, $SOS'O'$ is a harmonic range.

**Proposition 7.10.** The straight line joining the extremities of parallel radii of two circles passes through their external center of similitude, if they are turned in the same direction; through their internal center, if they are turned in opposite directions. And the converse.

(i) Let $O$, $O'$ be the centers of the $\odot$s; let $OX$ be $\parallel ZO'X'$ and let $XX'$ and $ZX$ meet $OO'$ in $S$ and $S'$ (Fig. 201). In $\triangle SO'X'$, $SOX$, $SO'/SO = O'X'/OX$. \[ S \text{ is the external center of similitude of the } \odot{s}. \] In $\triangle S'O'Z$, $S'OX$, $S'O'/S'O = O'Z/OX$. \[ S' \text{ is the internal center of similitude.} \]
Let $SXYX'Y'$ drawn through $S$ the external center of similitude of the circles cut the circles in $X, Y, X', Y'$. Then $OX$ will be || $O'X'$ and $OY$ || $O'Y'$. Since each of the $\angle OXY, O'X'Y'$ is acute, $\because$ each of the $\angle OSX, O'X'S$ is obtuse and $\angle OSX$ is common to $\triangle OSX, O'SX'$ and $SO : OX = SO' : O'X$. $\therefore \angle SOX = \angle SO'X'$. $\therefore OX$ is || $O'X'$. Similarly, $OY$ is || $O'Y'$. Similarly for the internal center of similitude.

**FIGURE 201**

**Corollary 1.** From similar triangles $SX : SX' = OX : O'X' = SY : SY'$. Therefore, the ratio of $SX : SX'$ is constant, as also the ratio of $SY : SY'$.

**Corollary 2.** If a variable circle touch two fixed circles both externally or both internally, the straight line joining the points of contact passes through the external center of similitude of the fixed circles.

**Corollary 3.** If a variable touches two fixed circles, one internally, the other externally, the straight line joining the points of contact passes through the internal center of similitude of the fixed circles.

**Corollary 4.** The direct common tangent of two circles passes through their external center of similitude, the transverse common tangent through their internal center of similitude.

**Definition.** Given a fixed point $O$ called the center of inversion. If on $OP$ a point $P'$ is taken such that $OP \cdot OP' = k$, where $k$ is a constant called the constant of inversion, then the points $P$ and $P'$ are said to be inverse points with respect to $O$.

If the constant $k$ is positive, the two points $P, P'$ lie on the same side of $O$ and if $k$ is negative, they lie on opposite sides of $O$. If $P, P'$ are on the same side $O$ so that $OP \cdot OP' = c^2$, then $P, P'$ are inverse points with respect to the circle with center $O$ and radius $c$. This circle is called the circle of inversion. The inverse of a point on the circle of inversion is the point itself; for in $OP \cdot OP' = c^2$, if $OP = c$, then $OP' = c$ and therefore $P'$ coincides with $P$. If $P$ describes a curve or a figure of any kind, then $P'$ describes another figure and the two figures are said to be inverse figures.
Notice that an intersection of two curves inverts into an intersection of the two inverse curves. Also, that if $OT$ touches the given curve at $T$, then $OT$ also touches the inverse curve at $T'$, the inverse of $T$. For, if $P$ and $Q$ coincide at $T$, $P'$ and $Q'$ coincide at $T'$.

**Proposition 7.11.** Two curves intersect at the same angle as the inverse curves. For this reason inversion is said to be a conformal transformation.

**Proposition 7.12.** If $P$, $P'$ and $Q$, $Q'$ are pairs of inverse points with respect to $O$, then $P'Q' = k \cdot PQ/OP \cdot OQ$, where $k$ is the constant of inversion.

For since $OP : OQ = OQ' : OP'$, the $\triangle s$ $OPQ$ and $OP'Q'$ are similar. Hence, $P'Q' : PQ = OQ' : OP = OP' : OQ' = k$. So $PQ = k \cdot P'Q'/OP' \cdot OQ'$. If, however, $P$, $Q$ are collinear with $O$, then $P'Q' = OQ' - OP' = k/OP - k/OQ = k(\text{OP} - \text{OQ})/OQ \cdot OP = k \cdot PQ/OP \cdot OQ$.

**Proposition 7.13.** The inverse of a straight line is a circle through the center of inversion $O$, and, conversely, the inverse of a circle through $O$ is a straight line.

(i) Draw $OA \perp$ the line and take any point $P$ upon the line. Let $A'$, $P'$ be the inverses of $A$, $P$ (Fig. 202). Then $OA \cdot OA' = OP \cdot OP'$. Hence, $A$, $A'$, $P'$, $P$ are concyclic. Hence $\angle OP'A' = \angle OAP = \text{right angle}$. Therefore, the inverse of the line $AP$ (i.e., the locus of $P'$) is the $\bigcirc$ on $OA'$ as diameter, $OA'$ being $\perp$ the line.

![Figure 202](image)

(ii) Take the inverse $A$, $P$ of $A'$, $P'$; $OA'$ being a diameter of the given $\bigcirc$. Then, as before, $\angle OAP = \angle OP'A' = \text{right angle}$. Hence the inverse with respect to $O$ of the circle on $OA'$ as diameter (i.e., the locus of $P$) is the straight line $\perp OA'$ through $A$. Therefore, $OC' = \frac{1}{2} OA' = \frac{1}{2} k/\text{OA}$. Hence the center $C'$ and the radius $OC'$ of the inverse to a given line are known.

An exceptional case is when the straight line passes through $O$, e.g., $OP$ in the above figure. Then $P'$ lies also on the line $OP$. Hence the inverse of a straight line through $O$ is the straight line itself. Notice that the inverse of a
circle with respect to a point \( O \) on it is a straight line. This property can be used mechanically to convert circular to linear motion.

It is obvious that we can invert any two circles which intersect into straight lines by taking \( O \) at one of the intersections. If the circles touch, their inverses would then be parallel straight lines.

**Proposition 7.14.** The inverse of a circle with respect to a point \( O \) not on it is a circle.

Let \( P' \) be the inverse of \( P \), a point on the given circle whose center is \( C \). Let \( OP \) cut this \( \odot \) again at \( Q \). Draw \( P'D' \parallel QC \) cutting \( OC \) at \( D' \) (Fig. 203). The \( OP \cdot OQ \) is constant and \( OP \cdot OP' \) is constant. Hence \( OP' : OQ \) is constant. But \( OD' : OC = OP' : OQ \). Hence \( OD' \) is constant and hence \( D' \) is fixed. Also, \( P'D' : QC = OP' : OQ \). Hence \( D'P' \) is fixed in length. Therefore, the locus of \( P' \) is a circle; i.e., the inverse of the given \( \odot \) is a \( \odot \).

In exactly the same way, it can be shown that the inverse of a sphere with respect to a point \( O \) not on it is a sphere.

**Proposition 7.15.** Find the center and radius of the inverse circle.

In the above figure, it can be seen that \( OD' / OC = P'D' / QC = OP' / OQ \) = \( OP \cdot OP' / OP \cdot OQ = k / (OC^2 - CP^2) \).

Hence \( OD' = k \cdot OC / (OC^2 - CP^2) \) and \( P'D' = k \cdot QC / (OC^2 - CP^2) \) or \( D' \) is the center of the inverse \( \odot \) whose radius is \( P'D' \).

**Proposition 7.16.** Inverse circles and the circle of inversion are coaxal.

For let the circle of inversion cut the given circle at \( E \) and \( F \). Then the inverse of \( E \) is \( E \). Hence the inverse circle passes also through \( E \) and similarly through \( F \).

As a particular case, a straight line is the radical axis of its inverse circle and the circle of inversion.

**Proposition 7.17.** The inverse of a circle with respect to an orthogonal circle is the circle itself.

For if \( O \) is the center and \( c \) the radius of the orthogonal circle and \( OPP' \) a chord (through \( O \)) of the given circle, then \( OP \cdot OP' = c^2 \), since a radius of one circle touches the other.
Proposition 7.18. Any two circles are inverse with respect to either center of similitude and with respect to no other point.

(i) In Fig. 203, let $O$ be a center of similitude of the circles, then $OP' : OQ = r' : r$. Also $OP \cdot OQ$ is constant. Hence, $OP \cdot OP'$ is constant. Hence the circles are inverse with respect to $O$.

(ii) Let the circles be inverse with respect to $O$. Then it is proved in Prop. 7.14 that $OC : OD' = r : r'$. Hence, $O$ is the center of similitude. Notice that the radius of inversion is $OE$, $E$ being one of the intersections of the given circles.

If in the limiting case one circle becomes a straight line, then we can still speak of the centers of similitude of this circle and this line. Hence, the centers of similitude of a circle and a line are the ends of the diameter perpendicular to the line, for these are the only possible centers of inversion.

Proposition 7.19. If a circle touches two given circles the points of contact are inverse points with respect to the external center of similitude.

For let the line joining the points of contact $P$, $Q$ cut the line of centers of the given circles $a$, $b$ at $O$. Invert with respect to $O$, taking $P$, $Q$ as inverse points. Then the touching circle $c$ inverts into itself. Also $a$ which touches $c$ at $P$ inverts into a circle touching $c$ at $Q$ and having, by symmetry, its center on the line of centers, i.e., into $b$. Hence $P$ and $Q$ are inverse points on the circles.

Solved Problems

7.16. If through the external center of similitude $S$ of two circles any straight line $SXYX'Y'$ cutting the first circle in $X$, $Y$ and the other in $X'$, $Y'$ be described, and also the common tangent $SDD'$, then each of the rectangles $SX$, $SY'$ and $SY$, $SX'$ is constant and equal to the rectangle $SD$, $SD'$.

Construction: Join $XD$, $YD$, $X'D'$, $Y'D'$ (Fig. 204).

![Figure 204](image)

Proof: $\therefore SX : SX' = OD : O'D' = SD : SD'$, $\therefore XD$ is $\parallel X'D'$. Similarly, $YD$ is $\parallel Y'D'$. $\therefore \angle SDX = \angle SD'X' = \angle D'Y'X'$. $\therefore DD'Y'X$ is cyclic. $\therefore SX \cdot SY' = SD \cdot SD'$. Similarly, $DD'X'Y$ is
cyclic. \[ SY \cdot SX' = SD \cdot SD' \]. Similar results can be proved in a similar way for the internal center of similitude.

**Corollary:** \( XD, Y'D' \) intersect in the radical axis of the two circles.

For since a circle goes through \( XDD'Y' \), therefore \( XD \) is the radical axis of this circle and the circle \( O \) and \( Y'D' \) is the radical axis of this circle and the circle \( O' \). Therefore, the point of intersection of \( XD, Y'D' \) is the radical center of the three circles and hence a point in the radical axis of the circles \( O, O' \).

**7.17. If two circles touch two others, so that the contacts of the two circles with the two others are both internally or externally or even one internally and the other externally, the radical axis of each pair passes through the center of similitude of the other pair (Fig. 205).**

**Proof:** Let \( \odot M, M' \) touch \( \odot O, O' \) externally in \( X, Y \) and \( X', Y' \). \( YX \) and \( Y'X' \) both pass through \( S \) the external center of similitude of \( O, O' \) (Prop. 7.10, Cor. 2). \[ SX \cdot SY = SX' \cdot SY'. \] Hence tangents from \( S \) to \( M \) and \( M' \) are equal. \[ S \] is on the radical axis of \( M, M' \). Similarly with \( S' \).

**7.18. A straight line drawn through a center of similitude \( S \) of two circles meets them in \( P, Q \) and \( P', Q' \) respectively. If \( PQ' \) is divided in \( R \) in the ratio of the radii of the circles, prove that the locus of \( R \) is a circle.**

**Construction:** Let \( PP'Q' \) be drawn from the external center of similitude of the two \( \odot O, O' \) and let \( S' \) be the internal center of similitude. Then the locus of \( R \) is the \( \odot \) on \( SS' \) as diameter (Fig. 206).
**Proof:** Join $OP$, $OQ$, $OR$, $O'P'$, $O'Q'$, $O'R$, $S'R$ and draw $SDD'$, $SEE'$ common external tangents of the circles. $OP$ is $\parallel O'P'$, $\therefore \angle OPR = \angle O'P'Q' = \angle O'Q'P'$. $\therefore PR : RQ' = OP : O'Q'$ (hypothesis), $\therefore PR : OP = RQ' : O'Q'$, and since $\angle OPR = \angle O'Q'P'$ or $\angle O'Q'R$, $\therefore \triangle ORP, O'Q'R$ are similar. $\therefore \angle ORP = \angle O'RQ'$ and $OR : O'R = OP : O'Q'$. $\therefore S'$ is the internal center of similitude of circles, $\therefore OS' : O'S' = OP : O'Q' = OR : O'R$. $\therefore \angle ORS' = \angle O'RPS'$, i.e., $RS'$ bisects $\angle ORO'$. Hence $\angle ORP + \angle ORS' = \text{right angle}; i.e., \angle SRS'$ is right. Since $SS'$ is fixed for the two circles, then the locus of $R$ is a circle on $SS'$ as diameter. Notice that this locus circle also cuts $SDD'$, $SEE'$ in $F$, $G$ in the ratio of the radii.

7.19. Peaucellier's Cell. $OB$, $OD$, $AB$, $AD$, $BC$, $CD$, $EA$ are rigid, very thin, straight rods, freely jointed at their ends as in the figure. $O$ and $E$ are fixed points. Also, $OB = OD$, $OE = AE$, and $AB = BC = CD = DA$. Show that as the rods move, $C$ describes a straight line (Fig. 207).
Proof: Since $EA = EO$, the locus of $A$ is a $\odot$ through $O$ with $E$ as center. Hence it is sufficient to prove that $A$ and $C$ are inverse points with respect to $O$, i.e., that $OA \cdot OC$ is constant. But $OA \cdot OC = (OM - MA)(OM + MA) = OM^2 - MA^2 = OB^2 - AB^2 = \text{constant}$. Therefore, $A$, $C$ are inverse points in the inversion whose $\odot$ of inversion has $O$ as center and radius equals $\sqrt{(OB^2 - AB^2)}$. Thus if $A$ describes a $\odot$, $C$ will describe the inverse of this $\odot$ which is a straight line $\perp OE$ from $C$ (Prop. 7.13). This mechanism, known as Peaucellier's cell, is useful in converting circular motion into straight-line motion. If we remove links $OE$ and $OA$, then as $A$ traverses any curve, $C$ will traverse its inverse curve. This cell, however, is sometimes called an inverter and is also used in the design of compound compasses for drawing arcs of circles with large radii.

7.20. Invert any triangle into a triangle of given shape.

Analysis: $\therefore A'B'/B'C' = (k \cdot AB \cdot OA \cdot OB)(OB \cdot OC \cdot k \cdot BC)$ (Prop. 7.12), $\therefore A'B'/B'C' = (AB/BC)(OC/OA)$. $\therefore A'B'/B'C'$ and $AB/BC$ are given from shape of $\triangle$, $\therefore OC/OA$ is a given ratio. Similarly, $OB/OA$ is also a given ratio. Hence $O$ is either one of intersections of two $\odot$s of Apollonius (Th. 5.10) drawn on $M'N'$ (on $AC$), and $MN$ (on $AB$) as diameters such that $OC/OA = M'C/M'A = N'C/N'A = \text{given}$ and so on ($O$, $O'$ are two inversion centers).

Synthesis: Draw a $\odot$ with $AB$ as chord cutting $OA$, $OB$ in $A'$, $B'$ such that $OA \cdot OA' = OB \cdot OB' = k$. Draw another $\odot$ around $\triangle ACA'$ cutting $OC$ in $C'$ such that $OA \cdot OA' = OC \cdot OC' = k$ (Fig. 208). Hence $A'B'C'$ is one of the two inverted $\triangle$s.

![Figure 208](image-url)
Miscellaneous Exercises

1. If $PL$, $PM$, $PN$, the perpendiculars from any point $P$ on the circumference of a circle to the sides $BC$, $CA$, $AB$ of an inscribed triangle $ABC$, be turned through the same angle about $P$ in the same direction and thus cut $BC$, $CA$, $AB$ in $L'$, $M'$, $N'$ then $L'$, $M'$, $N'$ are collinear. (They lie on a parallel to the pedal line turned through this angle, for $\frac{PL'}{PL} : \frac{PM'}{PM} = \frac{PN'}{PN}$.)

2. $P$, $Q$ are diametrically opposite points on a circle circumscribing a triangle. Perpendiculars from $P$ and $Q$ on their pedal lines with respect to the triangle meet at $R$. Show that $R$ is on the circle.

3. In Problem 7.1, prove that the circles $EBC$, $EDA$, $FAB$, $FDC$ meet in a point (i.e., the point $P$ of Problem 7.1.)

4. If $O$, $A$, $B$, $C$, $D$ are any concyclic points, show that the projections of $O$ on the pedal lines of $O$ with respect to the triangles $BCD$, $CDA$, $ABD$, $ABC$ are collinear.

5. $A$, $B$, $C$, $D$ are fixed points on a circle on which moves the variable point $P$. The pedal lines of $C$ and $D$ with respect to $ABP$ meet at $Q$. Find the locus of $Q$. (The pedal $x$ of $C$ with respect to $ABP$ passes through a fixed point, i.e., the projection of $C$ on $AB$, so the pedal $y$ of $D$ passes through a fixed point on $AB$. Also $x$, $y$ meet at an angle equal to $DAC$. Hence the locus of $Q$ is a circle.)

6. $A$, $B$, $C$, $D$ are concyclic points. Show that the angle between the pedal lines of $A$ with respect to $BCD$ and of $B$ with respect to $ACD$ is the angle subtended by $A$ and $B$ at the center of the circle.

7. Given the direction of the pedal line of $P$ with respect to a given triangle, find the position of $P$.

8. Prove the following construction for the pedal line $p$ of $P$: Bisect $PG$, $G$ being the orthocenter, in $P'$ and draw $p$ through $P'$ perpendicular to the line $AQ$ which is the reflection of $AP$ in the internal bisector of $BAC$.

9. Prove that the pedal lines of three points $P$, $Q$, $R$ on the circumference of $ABC$ with respect to $ABC$ form a triangle similar to $PQR$.

10. Lines drawn through a point $P$ on the circle $ABC$ parallel to $BC$, $CA$, $AB$ are turned about $P$ through a given angle in the same direction to cut $BC$, $CA$, $AB$ at $L$, $M$, $N$, $L'$, $M'$, $N'$ are formed in the same way from $P'$. Show that $LMN$, $L'M'N'$ are perpendicular lines if $PP'$ is a diameter of the circle. (By Exercise 1, $LMN$, $L'M'N'$ are $\parallel$ to the lines obtained by turning the pedal lines of $P$, $P'$ through a given angle. Hence the angle between them is right by Problem 7.2.

11. $P$ is a variable point on a given circle. $L$, $M$, $N$ are the projections of $P$ on the sides of a fixed inscribed triangle. $O_1$, $O_2$, $O_3$ are the centers of the circles $PMN$, $PNL$, $PLM$. Show that the circle $O_1O_2O_3$ is of fixed size.
12. $ABCD$ is a quadrilateral inscribed in a circle. Show that the pedal lines of $A$ with respect to $BCD$, of $B$ with respect to $CDA$, of $C$ with respect to $DAB$, and of $D$ with respect to $ABC$ meet in one point, and are equal to one another.

13. $P$ is a point on the circumcircle of the triangle $ABC$ and the perpendiculars from $P$ to $BC, CA, AB$ meet the circle again in $X, Y, Z$. Show that the perpendiculars from $A, B, C$ to the sides of $XYZ$ meet in a point $Q$ on the circle such that the pedal lines of $P$ and $Q$ with respect to $ABC$ and $XYZ$ respectively are coincident.

14. $P$ is any point on the circumscribed circle of a triangle $ABC$. $PL, PM, PN$ are the perpendiculars from $P$ to $BC, CA, AB$. If $PL, PM, PN$ are produced to $L', M', N'$ respectively, such that $PL = LL', PM = MM'$ and $PN = NN'$, show that $L', M', N'$ are collinear and pass through the orthocenter of the triangle $ABC$.

15. Find the locus of a point, given (a) the sum; (b) the difference of the squares on its tangents to two given circles.

16. In the triangle $ABC$, $AD, BE, CF$ are the altitudes on $BC, CA, AB$. $BC, EF$ meet at $P, CA, FD$ at $Q$ and $AB, DE$ at $R$. Show that $P, Q, R$ lie on the radical axis of the circumcircle and the nine-point circle.

17. If $A', B', C'$ bisect $BC, CA, AB$, show that each bisector of the angles of the triangle $A'B'C'$ is a radical axis of two of the circles touching the sides of $ABC$. (For a bisector of $A'$ is perpendicular to that bisector of $A$ which passes through the centers of two of the circles and $A'$ has equal tangents to these circles.)

18. A variable circle passes through two fixed points and cuts a fixed circle in the points $P$ and $Q$. Show that $PQ$ passes through a fixed point.

19. Pairs of points are taken on the sides of a triangle such that each two pairs are concyclic. Show that all six points lie on the same circle. (Otherwise the radical axes form a triangle.)

20. Prove that six radical axes of four circles, taken in pairs, form the six sides of a complete quadrangle. (For they meet, three by three, in four points.)

21. In the triangle $ABC$, prove that the orthocenter is the radical center of the circles on $BC, CA, AB$ as diameters.

22. The radical axes of a given circle and the circles of a coaxal system are concurrent. (Compare with Exercise 27.)

23. If $A, B, C$ are the centers and $a, b, c$ the radii of three coaxal circles, then $a^2 \cdot BC + b^2 \cdot CA + c^2 \cdot AB = BC \cdot CA \cdot AB$.

24. Through one of the limiting points of a system of coaxal circles is drawn a chord $PQ$ of a fixed circle of the system. Show that the product of the perpendiculars from $P$ and $Q$ on the radical axis is the same for all such chords. (Use Problem 7.6.)
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25. If a line cuts one circle at $P, P'$ and another at $Q, Q'$, show that $PQ$ and $P'Q'$ subtend, at either limiting point of their coaxal family, angles which are equal or supplementary. (For, if the line cuts the radical axis at $X$, $XL$ touches the circles $PLP'$ and $QLQ'$, since $XL^2 = XP \cdot XP' = XQ \cdot XQ'$. Hence in the case in which $PP'$ and $QQ'$ are on the same side of the radical axis, $\angle XLQ = \angle Q'LP$ and $\angle XLQ = \angle LP'P$. Hence, $\angle QLP = Q'LP'$. So also for the other case.)

26. If $PQ$ is a common tangent of two circles, then $PQ$ subtends a right angle at $L$ and $L'$, their limiting points. (For $XP = XQ = XL = XL'$.)

27. If two lines $AP$ and $AP'$ are divided at $Q, R, \ldots$ and $Q', R', \ldots$ so that $PQ : QR : \ldots = P'Q' : Q'R' : \ldots$, then the circles $APP', ARR', \ldots$ are coaxal.

28. Three circles 1, 2, 3 are such that the radical axes of 1, 2 and 2, 3 pass respectively through the centers of 3 and 1. Show that the radical axis of 3, 1 passes through the center of 2.

29. If the radical center of three circles is an internal point, show that a circle can be drawn which is cut by each of the three circles at the ends of a diameter of the circle.

30. If from any point in the radical axis of two circles a line be drawn to each circle, the four points of intersection lie on the circumference of a circle.

31. Prove that the orthocenter of a triangle is the radical center of the three circles described on the sides of the triangle as diameters.

32. Give a circle and a straight line $MN$. Describe a circle with given radius such that $MN$ shall be the radical axis of it and the given circle. (If $MN$ cuts the given circle, the two circles described with given radius through the points of intersection will be the required circles. If $MN$ touches the circle, the two circles described with given radius touching the given circle at the point of contact will be the required circles.)

33. Given a circle, a straight line $MN$ and a point $P$. Describe a circle passing through $P$ such that $MN$ shall be the radical axis of it and the given circle.

34. $A, B, C, D$ are four points in order in a straight line. Find a point $P$ in that line such that $PA \cdot PB = PC \cdot PD$. (This is the intersection of the radical axis of any two circles through $A, B$ and $C, D$ with the straight line $ABCD$.)

35. Prove that the radical axis of two circles is farther from the center of the larger circle than from the center of the smaller, but nearer its circumference.

36. Prove that if a circle be described with its center on a fixed circle and passing through a fixed point, the perpendicular from the fixed point on the common chord of the two circles will be of constant length.
37. From $A$ two tangents $AB$, $AB'$ are drawn to two circles. $AB$, $AB'$ are bisected in $D$, $D'$ and $DE$, $D'E$ drawn perpendicular to the lines joining $A$ and the centers of the circles intersect in $E$. Show that $E$ is on the radical axis of the circles.

38. $A$ is a fixed point on a given circle of a coaxal system whose limiting points are $L$ and $M$, and $P$ is any point on the tangent at $A$. Show that $AP^2 - PL^2 : AP^2 - PM^2 = AL^2 : AM^2$.

39. With given radius, describe a circle orthogonal to two given circles, $a$, $b$ with centers $A$, $B$. (Let the center of the required $O$ of radius $x$ be $X$; then $X$ lies on the radical axis of the given circles $a$, $b$. Let $XP$ be the radius of $x$ which touches $a$; then $x^2 = XP^2 = XA^2 - a^2 = XO^2 + OA^2 - a^2$. But $OA^2 - a^2$ is known and $x$ is given. Hence $OX$ and $X$ are known.)

40. Two variable circles pass through the fixed points $A$ and $B$. Through $A$ is drawn the line $PAQ$ to meet one circle at $P$ and the other at $Q$. Given that the angles $ABP$ and $ABQ$ are equal, find the locus of the intersection of the tangents at $P$ and $Q$.

41. $ABC$ is a triangle. $I_1$, $I_2$, $I_3$ are the centers of its inscribed and escribed circles. $a$, $S$, $y$, $b$ are the radical centers of these four circles, $b$ being that of the escribed circles. Show that the triangle $I_2I_3$ is similar to the triangle $I_1I_2I_3$ and that $b$, $a$, $S$, $y$ are the centers of the inscribed and escribed circles of the triangles formed by joining the middle points of the sides of $ABC$.

42. Construct a circle to pass through two given points $A$, $B$ so that its chord of intersection with a given circle shall pass through a given point $C$. (On the given circle take any point $P$ and let the $OABP$ cut the given circle again at $Q$. Let $PQ$, $AB$ meet at $R$, and let $RC$ cut the given circle at $X$, $Y$. Then $ABX$ is the required circle.)

43. If tangents from a variable point to two given circles have a given ratio, the locus of the point is a circle coaxal with the given circles. (Through any position $P$ of the point draw a circle $z$ with center $Z$ coaxal with the given circles $x$, $y$ (centers $X$, $Y$). Drop $PN \perp$ the radical axis of $x$ and $y$. Draw the tangents $PQ$ and $PR$ to $x$ and $y$. Then $PQ^2 = 2PN \cdot XZ$ and $PR^2 = 2PN \cdot YZ$. But $PQ : PR$ is constant. Hence $XZ : YZ$ is constant. Hence $Z$ is fixed. Therefore the coaxal through $P$ is fixed; i.e., the locus of $P$ is a circle coaxal with $x$ and $y$.)

44. A straight line meets two circles in four points. Show that the tangents at these points intersect in four points, two of which lie on a circle coaxal with the two given circles. (Use Exercise 43.)

45. $A$, $B$ are points on two circles $ACD$, $BEF$. It is required to find on the radical axis of these circles a point $P$ such that if the secants $PAC$, $PBE$ be drawn, $CE$ will be perpendicular to the radical axis.

46. From a point $P$ outside a given circle two straight lines $PAB$, $PCD$ are drawn, making equal angles with the diameter through $P$ and cutting
the circle in \( A, B \) and \( C, D \) respectively. Prove that \( AD, BC \) intersect in a fixed point. [Use Prop. 7.9(ii).]

47. Two fixed circles intersect and their common tangents intersect in \( F \). Prove that all circles which touch these given circles are intersected orthogonally by a circle of which \( F \) is the center. (Use Problem 7.14.)

48. Prove that the poles of the four rays of a harmonic pencil with respect to a given circle are collinear and form a harmonic range.

49. Prove that the polars of any point on a circle which cuts three circles orthogonally with respect to these circles are concurrent.

50. If \( A, B, C, D \) form a harmonic range, their polars \( a, b, c, d \) with respect to a given circle form a harmonic pencil. [For if \( AB \) is \( l \), \( a \) is the line through \( L \), the pole of \( AB \), \( \perp OA \). Hence, \( (a, b, c, d) \) is superposable to \( O(ABCD) \), \( O \) being the center of the circle.]

51. A circle touches the sides \( BC, CA, AB \) of a triangle at \( P, Q, R \) and \( RQ \) cuts \( BC \) at \( S \). Show that \( (BPCS) \) is harmonic. (For the polar of \( S \) passes through \( P \) and also through \( A \), since \( S \) is on the polar of \( A \).)

52. If \( P, Q \) are conjugate points with respect to a circle (see Prop. 7.7, Cor. 1), show that the circle on \( PQ \) as diameter is orthogonal to the given circle.

53. If \( P, Q \) are conjugate points with respect to a circle and \( t_1 \) and \( t_2 \) are the tangents from \( P, Q \) to the circle, show that the circles with centers \( P, Q \) and radii \( t_1, t_2 \) are orthogonal. (For \( PQ^2 = P'P'^2 + P'Q'^2 = t_1^2 - P'R^2 + P'Q'^2 = t_1^2 + t_2^2 \), where \( R \) is the chord of contact of \( P \).)

54. \( P, Q \) are conjugate points of a circle. \( U \) is the projection of the center of the circle on \( PQ \). Show that \( PU \cdot UQ \) is equal to the square of the tangent to the circle from \( U \).

55. The chord \( PQ \) is the polar of \( R \) with respect to a circle. \( S \) bisects any chord through \( R \). Show that \( SR \) bisects the angle \( PSQ \).

56. Two circles cut each other orthogonally. Show that the distances of any point from their centers have the same ratio as the distances of the centers each from the polar of the point with respect to the circle of which the other is the center.

57. The line joining any two of the six centers of similitude of three circles taken in pairs passes through a third center of similitude.

58. The internal and external centers of similitude of the circumcircle and the nine-point circle of a triangle are the centroid and the orthocenter.

59. Given two circles centers \( A, B \) and their circle of similitude described on the line joining their centers of similitude as diameter. Prove that if \( P \) is any point on the circle of similitude, then \( PA : PB \) is constant.

60. The circles of similitude of three circles taken in pairs are coaxal. (Let the centers be \( A, B, C \) and radii \( a, b, c \). Then if \( P \) is a point on the circles of similitude of the circles \( a, b \) and \( b, c \), then \( PA : PB = a : b \).
and \( PB : PC = b : c \) : : \( PA : PC = a : c \). Hence \( P \) is on the other circle of similitude.)

61. \( PQ, P'Q' \) are parallel chords of two circles the lengths of which are in the ratio of the radii. Show that \( PP', QQ' \) or \( PQ, P'Q \) meet at a center of similitude.

62. A common tangent to the circles the centers of which are \( A \) and \( B \) touches the circles at \( P \) and \( Q \). Show that \( PB \) and \( QA \) meet at a point which bisects the perpendicular from one of the centers of similitude to \( PQ \).

63. If the line joining the centers of two circles cuts the circles at \( B, C \) and \( B', C' \), show that the squares of the common tangents are equal to \( BB' \cdot CC' \) and \( BC' \cdot B'C \). [Draw \( \perp \) from center of smaller circles to radius of large circles at point of contact of a common tangent of length \( t \). If \( d, a, b \) are the distances between the centers and the radii, then \( t^2 = d^2 - (a \pm b)^2 \).]

64. \( t_1 \) and \( t_2 \) are the common tangents of two circles. Show that \( t_1^2 - t_2^2 = d_1d_2 \), where \( d_1 \) and \( d_2 \) are the diameters. (Use Exercise 63.)

65. The circles of similitude of three circles taken in pairs cut orthogonally the circle through the centers of the given circles.

66. \( AA', BB', CC' \) are the pairs of opposite vertices of a complete quadrilateral circumscribing a circle \( c \), \( A', B', C' \) being collinear. Show that half of the centers of similitude of the four circles \( ABC, AB'C', A'BC', A'B'C \), taken in pairs, lie on a line which bisects at right angles the line joining the center of \( c \) to the point through which the four circles pass.

67. Prove Problem 7.17 using inversion. (Invert with respect to \( S \) taking \( X, Y \) as inverse points. \( :: SX' \cdot SY' = SX \cdot SY = k, :: X', Y' \) are also inverse points, and \( \odot M, M' \) invert into themselves. Also \( \odot O \) inverts into \( \odot O' \). :: \( S \) is center of inversion and center of similitude of circles \( O, O' \).)

68. Two circles intersect at \( A \) and touch one line at \( P, Q \) and another line at \( R, S \). Show that the circles \( PAQ \) and \( RAS \) touch at \( A \). (Inverting with respect to \( A \), we get the parallel lines \( P'Q', R'S' \).)

69. The inverse \( C' \) of the center \( C \) of a circle is the inverse point of \( O \) with respect to the inverse circle. (See Prop. 7.14.)

70. Two circles are drawn to touch each of two given circles and also one another. Show that the locus of the point of contact of these two circles consists of the given circles when these do not intersect and the two coaxal circles which bisect the angles between the given circles when these intersect.

71. If tangents are drawn at \( A, B, C \) to the circumcircle forming the triangle \( LMN \), then the circumcircle of \( LMN \) belongs to the above system of coaxals. (Let \( OL \) cut \( BC \) in \( A' \). Then \( A' \) bisects \( BC \) and is the inverse of \( L \) with respect to the circumcircle.)
72. A circle orthogonal to two given circles cuts them in $A', B'$ and $C', D'$ and the line of centers cuts them in $A, B$ and $C, D$. Show that the lines $AA', BB', CC', DD'$ are concurrent for two locations of the points $A, B, C, D, A', B', C', D'$. (Invert the circles into themselves with respect to either intersection of the orthogonal circle and the radical axis.)

73. If three circles $c_1, c_2, c_3$ are such that $c_3$ is the inverse of $c_1$ with respect to $c_2$ and $c_1$ of $c_2$ with respect to $c_3$, then $c_2$ is the inverse of $c_3$ with respect to $c_1$.

74. A line is drawn through the fixed point $O$ to touch two circles of a given coaxal system at $P$ and $Q$. Find the locus of $R$, given that $(OR, PQ)$ is harmonic.

75. Two fixed circles touch internally at $A$. Show that the locus of the inverse point of $A$, with respect to a variable circle touching the given circles, is a circle whose radius is the harmonic mean between the radii of the given circles.
CHAPTER 8

SPACE GEOMETRY

Theorems and Corollaries

8.1. One part of a straight line cannot be in a plane and another part without.

8.2. Two straight lines which intersect are in one plane and three straight lines each of which cuts the other two are in the same plane.

Corollary 1. If three straight lines not in one plane intersect, two and two, they intersect in the same point.

Corollary 2. Similarly if four straight lines, no three of which are in the same plane, are such that each meets two of the others, they all meet in the same point.

8.3. If two planes intersect, their common section is a straight line.

8.4. If a straight line be perpendicular to each of two intersecting straight lines at their point of intersection, it is perpendicular to the plane in which they lie.

8.5. If three straight lines meet in a point and a straight line be perpendicular to each of them at that point, the three lines are in one plane.

8.6. Two straight lines which are perpendicular to the same plane are parallel.

8.7. If two straight lines are parallel, the straight line joining any point in the one to any point in the other is in the same plane with the parallels.

8.8. If two straight lines be parallel and one of them be perpendicular to a plane, the other is perpendicular to that plane.

8.9. Straight lines which are parallel to the same straight line, even though not in the same plane with it, are parallel to each other.

8.10. If two intersecting straight lines be respectively parallel to two other intersecting straight lines, though not in the same plane with them, the first two and the second two contain equal angles.

8.11. It is possible to draw a straight line perpendicular to a given plane from a given point outside the plane.

8.12. It is also possible to draw a straight line perpendicular to a given plane, from a given point in it.
8.13. Only one perpendicular can be drawn to a given plane from a given point, whether the point be in or outside the plane.
8.14. Planes to which the same straight line is perpendicular are parallel to each other.
8.15. If two intersecting straight lines in one plane be respectively parallel to two intersecting straight lines in another plane, the planes are parallel.
8.16. If two parallel planes be cut by any plane, their common sections with it are parallel.
8.17. If two straight lines be cut by parallel planes, they are cut in the same ratio.
8.18. If a straight line be perpendicular to a plane, every plane which passes through it is perpendicular to that plane.
8.19. If two planes which intersect be each of them perpendicular to a plane, their common section is perpendicular to that plane.
8.20. If a solid angle be contained by three plane angles, any two of them are together greater than the third.
8.21. Every solid angle is contained by plane angles which are together less than four right angles.

Corollary: There can be only five regular polyhedra.

(i) Three faces at least must meet to form each solid angle of a regular polyhedron.

(ii) The sum of the plane angles forming each of the solid angles is less than four right angles.

Now three angles of a regular hexagon are together equal to four right angles and three angles of any regular polygon of a greater number of sides are together greater than four right angles. Hence the faces of a regular polyhedron must be either equilateral triangles, squares, or regular pentagons.
8.21.1. If the faces are equilateral triangles, each solid angle of the polyhedron may be formed by: (i) Three equilateral triangles. The solid thus formed is a tetrahedron. (ii) Four equilateral triangles. The solid thus formed is an octahedron. (iii) Five equilateral triangles. The solid thus formed is an icosahedron. The angles of six equilateral triangles are together equal to four right angles and therefore cannot form a solid angle.
8.21.2. If the faces are squares, each solid angle will be formed by three squares. The solid thus formed is a cube. The angles of four squares are together equal to four right angles and cannot form a solid angle.
8.21.3. Similarly, if the faces are regular pentagons, each solid angle will be formed by three such pentagons. The solid thus formed is a dodecahedron.
8.22. The section of a sphere by a plane is a circle.
8.23. The curve of intersection of two spheres is a circle.
8.24. The properties of the radical plane, coaxal spheres, and the centers of similitude of two spheres follow by natural generalization from the corresponding concepts with circles in a plane.

8.25. If the radius of a sphere is \( r \), then the surface area and volume of the sphere are \( 4\pi r^2 \) and \( \frac{4}{3}\pi r^3 \) respectively.

8.26. Only one sphere can be drawn through four points which do not lie in a plane and no three of which lie on a line.

8.27. Eight spheres can, in general, be drawn to touch the faces of a tetrahedron.

8.28. If the vertices \( A, B, C, D \) of a tetrahedron are joined to the centroids \( A', B', C', D' \) of the opposite faces, the lines \( AA', BB', CC', DD' \) meet at a point \( G \) called the centroid of the tetrahedron such that \( AG = \frac{3}{4} GA' \), and so on.

8.29. If the radius of the base circle of a cone is \( r \) and its generator and altitude from the vertex to the base circle are \( l \) and \( h \), then the side surface area and volume of cone are \( \pi rl \) and \( \frac{1}{3}\pi r^2 h \) respectively.

8.30. Any surface area and volume of revolution of a plane figure are equal to the perimeter and area of that figure times the path of the center of area of the figure respectively: Pappus' theorem.

Solved Problems

8.1. \( AD \) and \( BC \) are two perpendiculars from the points \( A, B \) on a given plane. If a plane through \( A \) is drawn perpendicular to \( AB \) to cut the given plane in \( EF \), show that \( CD \) is \( \perp EF \).

**Construction:** Let \( DC \) and \( EF \) in the given plane intersect in \( G \). Join \( AG, AE \) (Fig. 209).
Proof: Since $AD, BC$ are $\perp$ given plane, \therefore they are $\parallel$ (Th. 8.6). Hence $A, B, C, D$ lie in one plane (Th. 8.7). \therefore $AB$ is $\perp$ plane $AGE$, \therefore plane $ABCD$ is $\perp$ plane $AGE$ (Th. 8.18). But since $AD$ is $\perp$ plane $GDE$, \therefore plane $AGD$ is $\perp$ plane $GDE$ (Th. 8.18). Therefore, plane $AGD$ is $\perp$ both planes $AGE$ and $GDE$. \therefore $GE$ is $\perp$ plane $AGD$ (Th. 8.19). \therefore $GE$ is $\perp$ any line in plane $AGD$ or $GE$ is $\perp$ $CD$; i.e., $CD$ is $\perp$ $EF$.

8.2. $CEDF$ is a plane and $A$ is a point outside it. $AC$ is perpendicular to the plane and $CD$ is perpendicular to $EF$ in the plane. Show that $AD$ is $\perp$ $EF$.

Construction: Join $AE, CE$ (Fig. 210).

\[ \begin{figure}[h] 
\centering 
\includegraphics[width=0.5\textwidth]{figure210.png} 
\caption{Figure 210} 
\end{figure} \]

Proof: Since $AC$ is $\perp$ plane $CEDF$, \because $AC$ is $\perp$ $CD$ and $CE$ (converse, Th. 8.4). Hence $AD^2 = AC^2 + CD^2$ and $AE^2 = AC^2 + CE^2$. \therefore Subtraction gives $AE^2 - AD^2 = CE^2 - CD^2$. But $CD$ is $\perp$ $EF$, \therefore $CE^2 - CD^2 = DE^2$. \therefore $AE^2 - AD^2 = DE^2$. \therefore $\angle ADE$ is right or $AD$ is $\perp$ $EF$.

8.3. Draw a straight line perpendicular to each of two straight lines not in the same plane. Prove that this common perpendicular is the shortest distance between the lines.

Construction: (i) Let $AB, CD$ be the given straight lines. Through any point $A$ in $AB$ draw $AE \parallel CD$. Draw $DF \perp$ plane $ABE$ and $FG \parallel AE$ meeting $AB$ in $G$. Make $DC$ equal to $FG$ and join $CG$ (Fig. 211).

Proof: \because $CD, FG$ are each $\parallel AE$, \therefore $CD$ is $\parallel FG$ and $CD = FG$ (construct). \therefore $CG$ is $\parallel DF$ and $DF$ is $\perp$ plane $ABE$. \therefore $CG$ is $\perp$ plane $ABE$. \therefore $CG$ is $\perp AB$ and $GF$ and $GF$ is $\parallel CD$. \therefore $CG$ is $\perp$ both $AD$ and $CD$. 


(ii) Draw any other straight line $DH$ between $AB$ and $CD$. Since $DF$ is $\perp$ plane $ABE$, $\therefore DF$ is $\perp FH$. $\therefore DH > DF$, i.e., $CG$.

8.4. Two parallel planes are cut by a straight line $ABCD$ in $B, C$ such that $AB = CD$. Two more transversals are drawn from $A$ and $D$ to cut the planes with $B$ and $C$ in $E, F$ and $G, H$. Show that $BEG$ and $CFH$ are two equal triangles (Fig. 212).

**Proof**: Plane $ACF$ cuts the two $\parallel$ planes. $\therefore BE$ is $\parallel CF$ (Th. 8.16). Similarly, $BG$ is $\parallel CH$. Hence $\angle EBG = \angle FCH$. Again, $AB/AC = BE/CF = CD/BD = CH/BG$. $\therefore BE/CF = CH/BG$. Therefore, the
\(\triangle EBG, FCH\) have both \(\angle s\) \(EBG, FCH\) equal and the sides about these equal \(\angle s\) inversely proportional. \(\therefore\) They are equal (Th. 4.94).

8.5. \(DABC\) is a tetrahedron with \(D\) as vertex and \(ABC\) as base. \(E, F, G\) are the middle points of \(BC, CA, AB\) respectively. If \(DF, DG\) are perpendiculars to \(AC, AB\) and \(BAC\) is a right angle, show that \(DE\) is perpendicular to the base \(ABC\).

Construction: Join \(AE, EG\) (Fig. 213).

![Figure 213](image)

**Proof:** Since \(F, G\) are the mid-points of \(AC, AB\) and \(DF, DG\) are \(\perp\) \(AC, AB\), then \(\triangle ADC, ADB\) are isosceles. Hence the edges \(DA, DB, DC\) of the tetrahedron are equal. \(\therefore \angle CAB = \text{right angle and } E\) is mid-point of \(CB, \therefore AE = EB\). Therefore, \(\triangle DEB, DEA\) are congruent. \(\therefore \angle DEB = \angle DEA = \text{right angle. Hence } DE\) is \(\perp\) the base plane \(ABC\) (Th. 8.4).

8.6. \(ABCD\) is a square and \(a\) is the length of its side. From \(A, C\) two perpendiculars \(AE, CF\) are drawn to the plane \(ABCD\) such that \(AE = AC\) and \(CF = \frac{3}{2} AC\). Find the length of \(BE, EF, BF\) in terms of \(a\). Then show that the triangle \(BEF\) is right-angled at \(E\) and that \(FE\) is perpendicular to the plane \(BDE\).

Construction: Join \(AC, FD\) and draw \(EG \perp FC\) (Fig. 214).

**Proof:** Since \(EA\) is \(\perp\) plane \(ABCD, \therefore EA\) is \(\perp AB\). \(\therefore BE^2 = AE^2 + AB^2\). But \(AE^2 = AC^2 = 2a^2\). \(\therefore BE^2 = 3a^2\). Hence \(BE = \sqrt{3}a\).

Again, \(CF\) is \(\perp\) plane \(ABCD\), i.e., \(\perp CA\) and \(\therefore EG\) is \(\perp FC\), \(\therefore EGCA\) is a square. \(\therefore EG = \text{and } || AC\). \(\therefore CF = \frac{3}{2} AC, \therefore GF = \frac{1}{2} AC\). \(\therefore EF^2 = FG^2 + GE^2 = \frac{1}{4} AC^2 + AC^2 = \frac{5}{4} AC^2 = \frac{5}{2} a^2\). \(\therefore EF = \sqrt{\frac{5}{2}}a\). Similarly, \(FC\) is \(\perp BC\). \(\therefore BF^2 = \frac{9}{4} \cdot 2a^2 + a^2 = \frac{11}{2} a^2\). \(\therefore BF = \sqrt{\frac{11}{2}}a\). Again, \(\therefore BE^2 + EF^2 = 3a^2 + \frac{5}{2} a^2 = \frac{11}{2} a^2 = BF^2\).
Therefore, $\triangle BEF$ is right-angled at $E$. Also, $\triangle EAB$, $\triangle EAD$ and $\triangle FCB$, $\triangle FCD$ are congruent. $\therefore EB = ED$ and $FB = FD$. $\therefore \triangle EBF$, $\triangle EDF$ are both congruent. $\therefore \angle BEF = \angle DEF = \text{right}$. $\therefore FE \perp ED$ and also $\perp EB$. $\therefore FE \perp \text{plane } BDE$ (Th. 8.4).

8.7. $ABCD$ is a tetrahedron in which the angles $ABC$, $ADC$ are right. If $M$, $N$ are the middle points of $BD$ and $AC$ respectively and $AM = MC$, show that $MN$ is the shortest distance between $AC$ and $BD$.

**Construction:** Join $NB$, $ND$ (Fig. 215).

**Proof:** Since $AM = MC$ (hypothesis), $\therefore \triangle AMN$, $\triangle CMN$ are congruent. $\therefore \angle ANM = \angle CNM = \text{right angle}$ or $MN$ is $\perp AC$. Again, since $\angle ABC = \angle ADC = \text{right angle}$ and $N$ is the mid-point
of $AC$ (hypothesis), .'. $BN = DN = \frac{1}{2} AC$. But $M$ is the mid-point of $BD$. .'. $\triangle BN M, DNM$ are congruent. .'. $\angle NMB = \angle NMD = \text{right angle}$. Hence $MN$ is also $\perp BD$. .'. It is the shortest distance between $AC, BD$.

8.8. From $D$, a fixed point outside plane $P$, a straight line $DE$ is drawn to cut the plane in $E$. $F$ is a point taken in $DE$ such that $DE - DF$ is constant. Find the locus of $F$.

**Construction:** Draw $DA \perp$ plane $P$. Join $EA$. In the plane $DAE$, draw $FB \perp DE$ to meet $DA$ in $B$. The locus of $F$ is a sphere on $DB$ as diameter (Fig. 216).

8.9. $ABCD$ is a parallelogram of plane paper in which $AD = 2 AB = 20$ inches and the $\angle A = 60^\circ$. $E, F, G$ are the middle points of $AD, BC, DC$ respectively and $AE, BE$ intersect in $M$; paper is folded about $BE$ so that the plane $ABE$ is perpendicular to the plane $BCDE$ (Fig. 217). Find the area of $\triangle AMG$.

**Proof:** Before paper is folded, $ABFE$ is a rhombus and $AF$ is $\perp BE$ at $M$; i.e., $AM$ is $\perp$ to the line of intersection of the perpendicular planes. Hence $AM$ is $\perp$ the plane $BCDE$. .'. $AM$ is $\perp MG$. .'. $\angle AMG = \text{right angle}$. Since $\triangle ABE$ is isosceles and $\angle A = 60^\circ$, .'. $\triangle ABE$ is equilateral. .'. $AM = \sqrt{10^2 - 5^2} = 5\sqrt{3}$ inches. But $MG = \frac{1}{2} (DE + BC) = 15$ inches. .'. $\triangle AMG = \frac{1}{2} (5\sqrt{3} \cdot 15) = 37.5\sqrt{3}$ inches$^2$. 
8.10. Find the total surface and volume of a regular tetrahedron and if these are \( \sqrt{2} \) ft\(^2\) and \( 2\sqrt{2}/3 \) ft\(^3\) respectively, find the corresponding lengths of the side.

Con**struction**: Let \( SABC \) be any regular tetrahedron having \( S \) as vertex and \( ABC \) as base. Draw \( SH \perp CB \) in the face \( \triangle CSB \). Draw also \( SD \) the altitude on the base \( ABC \), then join \( CD \) and produce it to \( E \) on \( AB \) (Fig. 218).

![Figure 217](image)

**Proof**: Total surface of tetrahedron = \( 2 \cdot BC \cdot SH = 2 \cdot BC(\sqrt{3}/2 \ BC) = \sqrt{3} \ BC^2 \). If this surface area = \( \sqrt{2} \) ft\(^2\), \( \therefore \) \( BC = \sqrt{2}/3 \) ft. Volume of tetrahedron = \( \frac{1}{3} \triangle ABC \times SD \). Similar to Th. 8.29, \( \because \) all sides are equal and \( CE \) is \( \perp \ AB \), \( \therefore \triangle ABC = \frac{1}{2} \ AB \cdot CE = a^2\sqrt{3}/4 \) \( (a \) being the side of tetrahedron). But \( D \) is the centroid of \( \triangle ABC \). \( \therefore \)
\[
CD = \frac{1}{3} CE = \frac{1}{3} a\sqrt{3}/2 = a/\sqrt{3}.
\]
Again, \( SD^2 = SC^2 - CD^2 = \frac{2}{3} a^2 \). \( \therefore \)
\( SD = a\sqrt{2}/3 \). Hence volume of tetrahedron = \( \frac{1}{3} a^2\sqrt{3}/4 \ a\sqrt{2}/3 = a\sqrt{2}/12 \). If this volume = \( 2\sqrt{2}/3 \) ft\(^3\), \( a = 2 \) ft.
8.11. ABC is a right-angled triangle at A. AD is drawn perpendicular to the hypotenuse BC. The triangle ADC is folded around AD to ADC' such that the angle between the planes C'AD and BAD is 60°. Show that the volume of the tetrahedron C'BAD = \(\frac{\sqrt{3}}{12}\) the volume of the cube with AD as side.

**Construction:** Let C be C' in the folded triangle C'AD. Draw C'E \(\perp\) plane ABD, to meet BD in E (Fig. 219).

**Proof:** Before folding \(\triangle ADC\), \(AD^2 = CD \cdot DB\). After folding, \(\therefore C'E \perp\) plane ABD, \(\therefore C'E \perp DE\).
\(\therefore \angle ADB = \text{right angle}, \therefore E\) is on \(DB\). \(\therefore \angle C'DE = 60°\).
\(\therefore C'E = CD \sin 60° = \sqrt{3}/2 \cdot CD\).

Volume of tetrahedron C'BAD = \(\frac{1}{3} \cdot \triangle DAB \times C'E = \frac{1}{3} (DB \cdot AD^2) / (\sqrt{3}/2) \cdot CD = \sqrt{3}/12 \cdot AD^3\).

8.12. AB, BC, CD are three sides of a cube the diagonal of which is AD. Show that the angle between the planes ABD, ACD is \(\frac{3}{4}\) of a right angle.

**Construction:** Draw BE, BF \(\perp\) to AD, AC. Join EF, BD (Fig. 220).
Proof: \( \therefore DC \) is \( \perp \) \( CB \), \( \therefore \) plane \( ADC \) is \( \perp \) plane \( ABC \). \( \therefore BF \) is \( \perp \) plane \( ADC \). \( \therefore BF \) is \( \perp \) \( FE \) (Th. 8.4). Let \( a \) be the side of the cube. \( \therefore AB, BD, AD \) are equal to \( a, \sqrt{2}a, \sqrt{3}a \) respectively. \( \therefore AB \) is \( \perp \) \( BD \). \( \therefore BE \cdot AD = AB \cdot BD = a^2 \sqrt{2} \). \( \therefore BE = a\sqrt{2}/3 \). Again, \( BF = a/\sqrt{2} \). \( \therefore \) sin \( FEB = BF/BE = (a/\sqrt{2})(\sqrt{3}/2)(1/a) = \sqrt{3}/2 \). \( \therefore \angle FEB = 60^\circ \), which is the angle between the planes \( ABD, ACD \).

8.13. If in a tetrahedron \( ABCD \) the directions of \( AB, CD \) be at right angles and also those of \( AC, BD \), so also will the directions of \( BC, AD \) be at right angles. Also the sum of the squares of each pair of opposite edges is the same and the four altitudes and the three shortest distances between opposite edges meet in the same point.

Construction: Draw \( AO \perp \) plane \( BCD \). Join \( BO, CO, DO \) and produce them to meet \( CD, DB, BC \) in \( E, F, G \). Join \( AE, AF, AG \) (Fig. 221).

![Figure 221](image)

Proof: \( \therefore BD \) is \( \perp \) \( AC \) and \( AO \) in the plane \( ACF \), \( \therefore BD \) is \( \perp \) \( AF, CF \). Similarly, \( CD \) is \( \perp \) plane \( ABE \) and \( \perp \) \( AE, EB \) in that plane. \( \therefore O \) is the orthocenter of \( \triangle BCD \). \( \therefore CB \) is \( \perp \) \( DG \) and \( CB \) is also \( \perp \) \( AO \). \( \therefore CB \) is \( \perp \) \( AD \). Since \( BD \) is \( \perp \) \( AF, CF \), \( \therefore AD^2 - AB^2 = DF^2 - BF^2 = CD^2 - CB^2 \). \( \therefore AD^2 + BC^2 = AB^2 + CD^2 \); also \( AC^2 + DB^2 \). Draw \( BK \perp \) plane \( ACD \). Then, as before, \( K \) is the orthocenter of \( \triangle ACD \) and \( \therefore \) lies in \( AE \), which is \( \perp \) \( CD \), as proved. \( \therefore BK \) meets \( AO \) in \( M \) the orthocenter of \( \triangle AEB \). Similarly, the \( \perp \) from \( C, D \) meet \( AO \), and hence it can be proven that any three of the four perpendiculars meet the other one. Hence they all intersect in the same point \( M \). Also the shortest distance between \( CD, AB \) is the \( \perp \) \( EH \) from \( E \) on \( AB \), for this is also \( \perp \) \( CD \). \( \therefore CD \) is \( \perp \) plane \( AEB \), \( \therefore EH \) passes through \( M \), the orthocenter of \( \triangle AEB \). Similarly, the other shortest distances pass through \( M \).
8.14. M is the vertex of a pyramid with the parallelogram ABCD as its base. In the face MBC, a line EF is drawn parallel to BC and cuts MB, MC in E, F. Show that AE, DF if produced will meet in a point the locus of which is a straight line parallel to the base ABCD (Fig. 222).

**Figure 222**

*Proof:* EF is || CB (hypothesis) and ABCD is a □. ∴ EF is || AD. ∴ EF is ≅ BC, ∴ also ≅ AD. Hence EF and AD form a plane of a trapezoid AEFD. ∴ If AE, DF are produced they will meet in a point N. In △AND, NF/ND = FE/DA. ∴ NF/ND = FE/CB = MF/MC. ∴ NF/FD = MF/FC. ∴ ∠MFN = ∠CFD, △s MFN, CFD are similar and MN is || CD and hence || to its plane ABCD. Since M is a fixed point and MN is || a fixed plane ABCD, ∴ the locus of N is a straight line through the vertex M and || plane ABCD.

8.15. A straight line AD cuts two parallel planes X, Y in B, C respectively such that AB : BC : CD = p : q : r. From A, D two other straight lines are drawn cutting planes X, Y in E, F and P, Q respectively. Show that

\[(rp + pq)CF \cdot CQ = (rp + rq)BE \cdot BP\] (Fig. 223).

*Proof:* AB/BC = p/q. ∴ AB/AC = p(p + q). ∴ Planes X, Y are parallel, ∴ BE is || CF (Th. 8.16). ∴ AB/AC = BE/CF = p(p + q).

∴ CF = \((p + q)/p)BE. Similarly, BP is || CQ and DC/DB = r(r + q) = CQ/BP. ∴ CQ = \((r + q)BP. Hence

\[CF \cdot CQ = \left(\frac{p + q}{p}\right)\left(\frac{r}{r + q}\right)BE \cdot BP\]

\[= \left(\frac{rp + rq}{rp + pq}\right)BE \cdot BP.\]
8.16. A, B, C, D are four points in space. The straight lines AB, DC; EF, GH are divided by points E, F; G, H such that AE:BE = DF:CF; AG:DG = BH:CH. Prove that the straight lines EF, GH lie in one plane.

Construction: From E draw the line C'ED' \parallel CD and from C, D the lines CC', DD' \parallel EF. Divide EF in M such that EM:MF = AG:GD = BH:CH. From G, H draw GG', HH' \parallel EF also meeting AD', BC' in G', H'. Join G'E, H'E, GM, HM (Fig. 224).

Proof: \therefore EM/MF = AG/GD, \therefore EF/EM = AD/AG = DD'/GG'.
But EF = DD'. \therefore EM = GG'. \therefore They are \parallel lines, \therefore EG' is \parallel MG.
Similarly, $EH'$ is equal to $MH$. \( \therefore DD', EF, CC' \) are parallel. \( \therefore ED'/EC' = DF/CF = AE/BE \) and $\triangle D'EA = \triangle BEC'$. \( \therefore \triangle D'EA, C'EB \) are similar. \( \therefore \angle D'AE = \angle C'BE \) and $AD'/BC' = AE/EB$. \( \therefore AG'/AD' = AG/AD = BH/BC = BH'/BC', \therefore AG'/BH' = AD'/BC' = AE/EB$. \( \therefore \angle EAG' = \angle EBH', \therefore \triangle AEC', BEH' \) are similar. \( \therefore \angle AE'G = \angle BE'H', \therefore \triangle AEC', BEH' \) are similar. \( \therefore AB \) is a straight line, \( \therefore G'H' \) is a straight line. Since $MG$ is parallel to $EG'$ and $MH$ is parallel to $EH'$, \( \therefore GMH \) is also a straight line. Therefore, $EF, GH$ intersect in $M$. \( \therefore \) They lie in one plane.

8.17. A prism has two equal and parallel base quadrilaterals $ABCD, A'B'C'D'$. A point $E$ is taken on $AD$ such that $DE = \frac{1}{3} AD$ and $D'C'$ is produced to $F$ such that $C'F = \frac{1}{3} D'C'$. Show that $A'C, EF$ intersect in a point and find the ratio in which $A'C$ is divided by this point.

**Construction:** Take $E'$ on $A'D'$ such that $D'E' = \frac{1}{3} A'D'$. Join $A'F, CE, C'E'$ (Fig. 225).

**Figure 225**

**Proof:** Since $DE = \frac{1}{3} DA$, also $D'E' = \frac{1}{3} A'D'$. \( \therefore CE \) is equal to $D'C'$ (since the bases of prism are equal and parallel). \( \therefore D'C' = 2 C'F. \therefore D'C'/D'E' = 3 = D'E'/A'D'. \therefore C'E' \) is parallel to $A'E' \| CE$. \( \therefore \) $A'C$ intersects $EF$ in $G$. In $\triangle D'A'F, D'C'/D'E' = 3 = A'E'/A'F = 3$. \( \therefore C'E' = CE, \therefore CE/A'E' = \frac{3}{2}. \therefore CE \) is parallel to $A'E' \| CG/AG' = CE/A'E' = \frac{3}{2}$.

8.18. $ABCD$ is a tetrahedron with $A$ as vertex and $AD$ is perpendicular to the base $BCD$. If $PQ$ is the shortest distance between the edges $AC, DB$ and $CR$ is drawn perpendicular to $DB$, show that $CP : PA = CR^2 : DA^2$.

**Construction:** From $R$ draw $RS$ equal to $AD$. Join $AS, CS$ and draw $PT \| AS$ to cut $CS$ in $T$. Join $RT$ (Fig. 226).

**Proof:** \( \therefore \) $QP$ is perpendicular to $AC$ at $P$ (hypothesis) \( \therefore \) $ASRD$ is a parallelogram, i.e., $AS \| DR \| PT$ and also $PQ$ is perpendicular to $BR$ at $Q$. \( \therefore \) $PQ$ is also perpendicular to $PT$. \( \therefore \) $PQ$ is perpendicular to the plane $PTC$ (Th. 8.4). But $BR$ is perpendicular to $RS$ and $RC$. 


hence \( \perp \) plane SRC. \( \therefore \) BR is \( \perp \) RT. \( \therefore \) BR is \( \parallel \) PT, \( \therefore \) PT is \( \perp \) RT.

\( \therefore \) PQRT is a rectangle. Hence PQ is \( \parallel \) RT. \( \therefore \) PQ is \( \perp \) plane PTC, \( \therefore \) also RT is \( \perp \) plane PTC. \( \therefore \) RT is \( \perp \) TC. \( \therefore \) SR is \( \parallel \) AD and hence \( \perp \) plane BRC or \( \perp \) RC. \( \therefore \) \( \triangle \) SRC is right-angled at R and RT is \( \perp \) hypotenuse SC or TC. \( \therefore \) \( CR^2 = CT \cdot CS \) and \( SR^2 = ST \cdot CS = AD^2 \).

\( \therefore \) \( CR^2/AD^2 = CT/ST = CP/PA \).

8.19. A cube is constructed inside a right circular cone with vertex A such that four of its vertices lie on the cone surface and the other four vertices on the base of the cone. If the ratio between the altitude of the cone and its base radius is \( \sqrt{2} : 1 \), show that the side of the cube = \( \frac{1}{\sqrt{2}} \) the cone altitude.

**Construction:** Let the cube be \( DEFGHLMN \) of side a. Draw the altitude AO of the cone on the base cutting the upper face of the cube in O'. Join LN and produce it to meet the base circumference in B, C (Fig. 227).
Proof: Since the cone is right and cube vertices lie on its surface and base, then $O, O'$ are the centers of the faces $DEFG, HLMN$ of the cube and hence $EO'G, LON$ are diameters of the upper circular section and respectively $\parallel$ and coplanar to the base. \[AO : BO = \sqrt{2} : 1\] and $EG$ is $\parallel BC, \therefore AO/BO = AO'/EO' = \sqrt{2} \therefore AO' = \sqrt{2} E0'$. But $AO' = AO - OO' = AO - a$ and $EO' = \frac{1}{2} EG = \sqrt{2}/2 a$. \[\therefore AO' = \sqrt{2} E0' = a = AO - a. \therefore a = \frac{1}{2} AO.\]

8.20. $ABCD$ is a tetrahedron and $M, L, R$ are three points on the edges $AB, AC, AD$. If the lines $MR, ML, LR$ are produced to meet the sides of the base $BD, BC, CD$ or produced in $X, Y, Z$, show that $XYZ$ is a straight line: Desargue's theorem (Fig. 228).

![Figure 228](image)


8.21. $MA, MB, MC$ are three edges of a cube meeting in the vertex $M$. If $a$ denotes the side of the cube, show that (i) volume of tetrahedron $MABC = a^3/6$; (ii) area of triangle $ABC = a^2\sqrt{2}/2$; (iii) the perpendicular from $M$ on plane $ABC = a\sqrt{3}/3$.

Proof: (i) Volume of tetrahedron $MABC$ is the same as volume of tetrahedron, $CMBA = \frac{1}{3} MC \cdot \triangle AMB = (a/3)(a^2/2) = a^3/6$.

(ii) \[\therefore MD \parallel AB, \therefore\] it bisects $AB$ in $D$. Again $BC = AC = face$ diagonal. Hence $ABC$ is an equilateral $\triangle$. \[\therefore CD$ is $\perp AB$ (since $D$ is the mid-point of $AB$). \[\therefore AB = a\sqrt{2}, MD = a/\sqrt{2}$, and $CM$ is $\perp MD, \therefore CD^2 = MC^2 + MD^2 = a^2 + (a^2/2) = \frac{3}{2} a^2. \therefore CD$
= a(\sqrt{3}/2). \text{Hence area of } \triangle ABC = \frac{1}{2} \ AB \cdot CD = (a\sqrt{2}/2)(a\sqrt{3}/\sqrt{2}) = a^2\sqrt{3}/2.

(iii) \therefore \text{Volume of tetrahedron } MABC = \frac{1}{3} ME \cdot \triangle ABC (ME \text{ is } \perp \text{plane } ABC), \therefore a^3/6 = \frac{1}{3} ME(a^2\sqrt{3}/2). \therefore ME = a/\sqrt{3} = a\sqrt{3}/3.

8.22. If the area of the side surface of a right circular cone is equal to \frac{1}{6} of the area of the circle having as radius the generator of the cone, show that the ratio between the volume of the cone and that of a sphere which passes through the vertex and base circle of the cone is 225 : 2048.

CONSTRUCTION: Let ABC be the right circular cone with A as vertex and BC a diameter in its base. Draw its altitude AD and produce it to meet the sphere in E. Join CD, CE (Fig. 230).
Proof: Let the altitude $AD = h$, generator $AB = l$, radius of base $O = r$ and radius of sphere $= R$. \[ \therefore \text{Area of cone side surface} = \pi rl = \frac{1}{3} \pi l^2 \text{(hypothesis)}, \therefore l = 4r. \therefore AD \perp \text{base } O, \therefore \perp CD. \therefore AC^2 = AD^2 + CD^2 \text{ or } l^2 = h^2 + r^2. \therefore h = r\sqrt{15}, \therefore \text{volume of cone} = \frac{(\pi/3)r^2h = (\pi/3)r^3\sqrt{15}}{2} \text{. Since } AE \text{ is a diameter of the sphere, } \therefore \angle ACE \text{ is right. } \therefore CD \text{ is } \perp \text{AE}, \therefore l^2 = AD \cdot AE \therefore AE = \frac{l^2}{h} = 16r/\sqrt{15}. \therefore R = 8r/\sqrt{15}. \text{Volume if sphere} = \frac{4}{3} \pi R^3 = \frac{4}{3} \pi (512 \times 15/\sqrt{15}) = 2048 \pi r^3/45/\sqrt{15}. \therefore \text{Volume of cone} : \text{volume of sphere} = \frac{(\pi/3)r^3\sqrt{15}}{2048 \pi r^3/45/\sqrt{15}} = 225 : 2048.

8.23. MA, MB, MC are three straight lines in space; each is perpendicular to the other two lines. If $x$, $y$, $z$ are the lengths of these lines respectively, show that (i) volume of tetrahedron $MABC = \frac{1}{6} xyz$; (ii) area of the triangle $ABC = x^2y^2 + 2z^2 + z^2x^2$; (iii) the altitude from $M$ on the plane $ABC = xyz/xyz + y^2z^2 + z^2x^2$.

Construction: Draw $MD \perp BC$ then join $AD$. Let $MN$ the altitude from $M$ on $ABC$ be denoted $l$ (Fig. 231).

\[ \text{Figure 231} \]

\[ \text{Proof}: \] (i) Volume of tetrahedron $= \frac{1}{3} AM \cdot \triangle MBC = \frac{1}{6} AM \cdot MB \cdot MC = \frac{1}{6} xyz$.

(ii) $\therefore AM$ is $\perp$ plane $MBC$ and $MD \perp BC, \therefore AD$ is $\perp BC$ also (Th. 8.4). In $\triangle MBC, \angle M$ is right. $\therefore MB \cdot MC = 2 \triangle MBC = MD \cdot BC$. $\therefore MD = MB \cdot MC/BC = yz/BC$. $\therefore AD^2 = AM^2 + MD^2 \therefore AD = \sqrt{AM^2 + MD^2} = \sqrt{x^2 + (y^2z^2/BC^2)} = (1/BC)\sqrt{x^2 \cdot BC + y^2z^2}. \text{But } \triangle ABC = \frac{1}{2} AD \cdot BC = \frac{1}{2} \sqrt{x^2 \cdot BC + y^2z^2}. \therefore BC^2 = y^2 + z^2, \therefore \triangle ABC = \frac{1}{2} \sqrt{x^2y^2 + x^2z^2 + y^2z^2}.

(iii) $\therefore \text{Volume of tetrahedron} = (l/3) \triangle ABC, \therefore \frac{1}{6} xyz = l/6 \sqrt{x^2y^2 + y^2z^2 + z^2x^2}. \therefore l = xyz/\sqrt{x^2y^2 + y^2z^2 + z^2x^2}.

Also $\therefore 1/l^2 = (x^2y^2 + y^2z^2 + z^2x^2)/x^2y^2z^2 = 1/x^2 + 1/y^2 + 1/z^2$. 
This result is also true in a two-dimensional problem. If \( l \) is the length of the altitude from the right vertex on the hypotenuse \( C \) and \( x, y \) are the lengths of the sides surrounding the right angle in any right-angled triangle, then \( xy = cl \). \[ \therefore x^2 y^2 = c^2 l^2 = (x^2 + y^2)l^2. \]

8.24. If \( V \) denotes the volume formed by the rotation of a right-angled triangle around its hypotenuse and \( V_1, V_2 \) the volumes formed by the rotation of the triangle around the sides of the right angle, show that \( 1/V^2 = 1/V_1^2 + 1/V_2^2 \).

**Construction:** Draw from the right vertex \( B \) of \( \triangle ABC \), \( BD \perp AC \). Let \( a, b, c \) be the lengths of the sides \( BC, CA, AB \), and \( BD = h \) (Fig. 232).

![Figure 232](image)

**Proof:** \( V \) = the volume of the two cones formed by the rotation of \( \triangle ADB, CDB \) around \( AD, CD = (\pi/3)h^2b \) and \( V_1 = \) volume of cone formed by rotation around \( AB = (\pi/3)a^2c \). Also, \( V_2 = \) volume of cone around \( BC = (\pi/3)c^2a \). It is required to show that \( V_1^2 V_2^2 = V_2(V_1^2 + V_2^2) \) or \( a^4 c^2 c^4 a^2 = a^6 c^6 = h^4 b^2 (a^4 c^2 + c^4 a^2) \). But \( h^4 b^2 (h^2 b^2 a^2 + h^2 b^2 c^2) = h^6 b^4 (a^2 + c^2) = h^6 b^6 = a^6 c^6 \).

8.25. On the same circular base of a hemisphere with center \( O \), a frustum of a cone is constructed inside the hemisphere, the bases of which are parallel. If the ratio between the radii of the cone bases is 1 : 2, show that the ratio of the side surface area of the cone to the hemispherical surface area is 3 : 4.

**Construction:** Let \( AB, CD \) be two \( \parallel \) diameters in the bases of the frustum, \( AB \) being the common diameter with the hemisphere. Join \( DO \) and draw \( CE \perp AB \). Let \( OB = r \) (Fig. 233).

**Proof:** \( \therefore CD = \frac{1}{2} AB = BO = OD \) (hypothesis) and \( CD \parallel BO \), \( \therefore OBCD \) is a rhombus in which \( CO = CB \). \( \therefore CE \) bisects \( BO \) in \( E \). Since side surface area of cone frustum = \( \frac{1}{2} BC(\pi \cdot AB + \pi \cdot CD) = 3 \pi r^2/2 \) and hemispherical surface area = \( 2 \pi r^2 \), the ratio between their surface areas = 3 : 4.
8.26. If a sphere with radius $r$ touches internally a right circular cone whose altitude and base radius are $h$, $R$, show that $\frac{1}{r^2} = \frac{1}{R^2} + \frac{2}{rh}$. If $V$, $S$ and $V_1$, $S_1$ are the volume and surface of sphere and cone respectively, show also that $V : V_1 = S : S_1$.

**Construction:** Let $BC$ be a diameter in the base of that cone and $A$ be the vertex. Draw $AO \perp BC$, $O$ being the base center. If $M$ is the center of the sphere which touches the cone at $E$, join $ME$ (Fig. 234).

**Proof:** \[ AE^2 = AM^2 - ME^2 = AM^2 - MO^2 = AO^2 - 2 \cdot AO \cdot MO = h^2 - 2 \cdot hr \] and \[ \triangle AME, ACO \text{ are similar, } \therefore \frac{AE}{AO} = \frac{EM}{CO}. \] \[ \therefore \frac{(h^2 - 2 \cdot hr)}{h^2} = \frac{r^2}{R^2} \text{ or } 1 - \frac{2r}{h} = \frac{r^2}{R^2}. \]

Dividing by $r^2$ and rearranging, \[ \therefore \frac{1}{r^2} = \frac{1}{R^2} + \frac{2}{hr}. \] Join $MC$.

Since $MC$ bisects $\triangle ABC$ in plane $ACB$, \[ \therefore \frac{CO}{AC} = \frac{MO}{AM}. \] Let the generator $AC = l$. \[ \therefore \frac{R}{l} = \frac{r(h - r)}{Rh} = \frac{r(l + R)}{Rh}. \] \[ \therefore V : V_1 = \frac{4}{3}\pi r^3 : (\pi/3)R^2 h = 4\pi r^2 : \pi R(l + R). \] \[ \therefore S : S_1 = 4 \pi r^2 : \pi R(l + R) = V : V_1. \]

8.27. If the inclined generator of a frustum of a cone is equal to the sum of the radii of the bases, show that the altitude = twice the geometric mean of these radii, and the volume = the entire surface of the frustum times $\frac{1}{6}$ its altitude.
Construction: Take a projection of the cone frustum $ABCD$ in which $AB = 2R$ and $CD = 2r$. Draw $CM \perp AB$ and let the generator $AD = CB = l$, and the altitude $= h$ (Fig. 235).

![Figure 235](image)

**Proof:** In $\triangle CBM$, $CM = h$. \[ \therefore CB^2 = CM^2 + BM^2 \text{ or } h^2 = l^2 - (R - r)^2. \] \[ \therefore l = R + r \text{ (hypothesis)}, \therefore h^2 = (R + r)^2 - (R - r)^2 = 4Rr. \] Therefore, $h = 2\sqrt{Rr}$. Again, entire surface of frustum $= (l/2)(2\pi R + 2\pi r) + \pi R^2 + \pi r^2$. Entire surface $\times \frac{h}{6} = \frac{(h/6)\pi[(R + r)^2 + R^2 + r^2]}{6} = \frac{(\pi h/3)(R^2 + r^2 + Rr)}{6}$. \[ \therefore \text{The volume of this frustum} = \frac{(h/3)(\pi R^2 + \pi r^2 + \sqrt{\pi R^2 \cdot \pi r^2})}{6} = \frac{(\pi h/3)}{6} \times \frac{(R^2 + r^2 + Rr)}{6} = \text{entire surface} \times \frac{h}{6}.

8.28. On $AOB$ as diameter and $O$ as center a circle is described and two points $D, E$ are taken on $AB$ such that $OD \cdot OE = OB^2$. On $DE$ as diameter a semi-circle is described such that its plane is perpendicular to the plane of the circle $O$. If any points $R, L$ are taken on the circle $O$ and $X, Y$ on the semi-circle $DE$, show that $RX : RY = LX : LY$.

**Construction:** Draw $RP, LQ, XF, YS \perp ABE$. Join $LS$. Let $M$ be center of the semi-circle (Fig. 236).

![Figure 236](image)

**Proof:** Since $OD \cdot OE = OB^2$, then, if the semi-circle plane is rotated a right angle to coincide with the plane of $O$, they will cut.
orthogonally. If the radii of $\bigcirc O$ and semi-circle are $r, r_1$, then $OM^2 = r^2 + r_1^2$. \( \therefore \) $YS$ is $\perp AB$, and planes of $\bigcirc s$ are $\perp$ each other. \( \therefore \) $YS$ is $\perp$ plane $\bigcirc AOB$. \( \therefore \) $YS$ is $\perp SL$ which lies in the plane of $\bigcirc O$. But $LY^2 = YS^2 + SL^2 = LQ^2 + QS^2 + YS^2 = r^2 - QO^2 + QS^2 + r_1^2 - MS^2 = OM^2 - QO^2 + QS^2 - MS^2 = 2 QM \cdot OS$ and $LX^2 = 2 QM \cdot OF$. \( \therefore \) $LX^2 : LY^2 = OF : OS$. Similarly, $RX^2 : RY^2 = OF : OS$. Hence $RX : RY = LX : LY$.

8.29. $ABCD_A1B_1C_1D_1$ is a given regular prism, the faces of which are parallelograms and $P$ is a fixed plane. Find the locus of the point $E$ in the plane $P$ such that the sum of the squares on its distances from the vertices of the prism is constant.

**Construction:** Join the diagonals $AC_1$, $A_1C$, $BD_1$, $B_1D$, which intersect in one point $M$ (since the figure is a regular prism). Draw $MO \perp$ plane $P$ and join $ME, OE$ (Fig. 237).

**Proof:** Since $M$ is the middle point of all the diagonals of prism, \( \therefore \) $EA^2 + EC_1^2 = 2 EM^2 + 2 MA^2$. Similarly with the other three pairs of diagonally opposite vertices. Adding up gives $8 EM^2 = EA^2 + EB^2 + EC^2 + ED^2 + EA_1^2 + EB_1^2 + EC_1^2 + ED_1^2 - 2 (MA^2 + MA_1^2 + MB^2 + MB_1^2)$. Assuming that the sum of the squares of the distances from $E$ to the vertices $= S$ and since $EM^2 = MO^2 + OE^2$, \( \therefore \) $8 (MO^2 + OE^2) = S^2 - 2 (MA^2 + MA_1^2 + MB^2 + MB_1^2)$. \( \therefore \) $OE^2 = \frac{1}{8} \{S^2 - 2 (MA^2 + MA_1^2 + MB^2 + MB_1^2 + 4 MO^2)\} = \text{constant}$. \( \therefore \) The projection $O$ of the fixed point $M$ on the fixed plane $P$ is fixed, the locus of $E$ is a $\bigcirc$ in plane $P$ with $OE$ as radius.
8.30. The total surface area of a cylinder whose diameter is equal to its altitude and which is constructed outside (inside) a sphere is the mean proportional between the surface area of this sphere and the total surface area of the right circular cone whose generator is equal to the diameter of its base and which is also constructed outside (inside) the same sphere. Prove that this is also the case with their volumes.

Construction: Let the cylinder and cone be described outside the sphere with center $M$. Let the sphere touch the base and surface of cone in $D, E$. Join $AD, ME$. $R, r$ denote the radii of cone base and sphere (Fig. 238). Let the shown projection represent the solids in question. Total area of cone = $\pi Rl + \pi R^2$. But generator $l = 2R$. So, total area of cone = $3\pi R^2$. Area of sphere = $4\pi r^2$. Since cone $ABC$ has $\angle A = 60^\circ$, $\therefore ME/AE = r/R = \tan 30^\circ$. $\therefore r = R/\sqrt{3}$. Area of sphere = $\frac{4}{3}\pi R^2$. If cylinder altitude = $h$, total area of cylinder = $6\pi r^2 = 2\pi R^2$. Therefore area of sphere × total area of cone = $3\pi R^2 \cdot \frac{4}{3}\pi R^2 = 4\pi^2 R^4 = (\text{total area of cylinder})^2$. Again, volume of cone = $\frac{1}{3}\pi R^2 H$. But $H$ the altitude of cone = $R/\sqrt{3}$. So, volume of cone = $(\sqrt{3}/3)\pi R^3$. Volume of sphere = $\frac{4}{3}\pi r^3 = 4\pi R^3/9\sqrt{3}$. Volume of cylinder = $\pi r^2 h = 2\pi r^3 = 2\pi R^3/3\sqrt{3}$. Hence volume of sphere × volume of cone = $4\pi^2 R^6/27 = (\text{volume of cylinder})^2$. Similarly for the case when the cylinder and cone are described inside the sphere.

Miscellaneous Exercises

1. Given three non-intersecting straight lines in space, draw from any point in one a straight line intersecting both the others. Examine the cases in which more than one line can be so drawn, also the cases in which no line can be so drawn.
2. Draw a plane perpendicular to a given straight line through a given point either in the line or without it.

3. If a pyramid stands on a square base and has equilateral triangles for its faces, prove that the perpendicular from the vertex to the base is equal to half the diagonal of the base.

4. A piece of wire is bent into three parts, such that each of its exterior parts is double the middle one and perpendicular to the plane containing the other two of which the middle part is one. Show that the line joining the ends of the wire = three times the middle part.

5. Of all the straight lines which can be drawn to meet a given plane from a given point outside it, the least is the straight line perpendicular to the plane, and those which meet the plane in points equally distant from the foot of the perpendicular are equal, and of two straight lines the one which meets the plane at a point farther from the foot of the perpendicular than another is the greater and conversely.

6. $AB$ is a straight line. From any point $A$ on the line, $AC, AD$ are drawn perpendicular to $AB$, and from another point $B$ in the line $BE, BF$ are drawn perpendicular to $AB$. Show that the planes $ACD, BEF$ are parallel.

7. $AB, CD$ are two parallel lines and from any point $E$ outside their plane, $EF$ is drawn perpendicular to $AB$ meeting $AB$ in $F$ and $FR$ drawn perpendicular to $CD$ meeting $CD$ in $R$. Show that $ER$ is perpendicular to $CD$.

8. $MA, MB$ are two straight lines intersecting in $M$ and $MC$ is another straight line not in their plane such that $\angle AMC = \angle BMC$. From any point $C$ in $MC$, $CD$ is drawn perpendicular to the plane $MAB$. Show that $DM$ bisects the angle $AMB$. (From $C$ draw $CE, CF$ perpendiculars to $MA, MB$ and join $DE, DF$.)

9. From the point $P$ outside two intersecting planes in $AB$, two perpendiculars $PQ, PR$ are drawn to the planes. If $QS$ is also drawn perpendicular to the plane where $R$ lies, then prove that $RS$ is perpendicular to $AB$.

10. Find the locus of points equally distant from two given points.

11. $AB, AC$ are two perpendiculars from any point $A$ on two intersecting planes in $DE$. Show that $BC$ is perpendicular to the parallel from $C$ to $DE$.

12. $ABC$ is a triangle and $M$ is the orthocenter. If $MP$ is drawn perpendicular to the plane of the triangle, show that the line joining $P$ to any one of the vertices of the triangle will be perpendicular to the parallel from that vertex to the opposite side.

13. Find the locus of points equally distant from three given points.

14. Given a straight line and any two points, find a point in the straight line equally distant from the two points.
15. Prove that the sides of an isosceles triangle are equally inclined to any plane through the base.

16. A given line $l$ is parallel to a given plane. $x$ is a variable line in this plane. Show that unless $x$ is parallel to $l$, the shortest distance between $x$ and $l$ is constant.

17. Draw a straight line to meet three given straight lines and be parallel to a given plane. (Let the lines be $AB$, $CD$, $EF$. Draw through $CD$ a plane $\parallel$ to given plane. Let $AB$, $EF$ cut this plane in $P$, $Q$. Then $PQ$ is such a line.)

18. Find the locus of the foot of the perpendicular from a given point on a plane which passes through a given straight line.

19. From a point $P$, $PA$ is drawn perpendicular to a given plane, and from $A$, $AB$ is drawn perpendicular to a line in that plane. Prove that $PB$ is also perpendicular to that line.

20. $OA$, $OB$, $OC$ are three straight lines mutually perpendicular. From $O$ perpendiculare $OP$, $OQ$, $OR$ are let fall on $BC$, $CA$, $AB$. Prove that $BC$ bisects the angle $QPR$ externally.

21. $OP$ is at right angles to the lines $OA$, $OB$, which are also at right angles to each other. $OC$, $OD$ are drawn bisecting the angles $POA$, $POB$. Prove that $COD$ is $60^\circ$. (Make $OC = OD$, and draw $CA$, $DB \perp OA$, $OB$.)

22. From a point $P$, $PA$, $PB$ are drawn perpendicular to two planes which intersect in $CD$ meeting them in $A$ and $B$. From $A$, $AE$ is drawn perpendicular to $CD$. Prove that $BE$ is also perpendicular to $CD$.

23. Prove that the intersection of two planes, each of which contains one of two parallel straight lines, is parallel to those lines.

24. Planes are drawn through a given point each containing one of a series of parallel straight lines. Prove that they intersect any plane which intersects their common line through the given point in three concurrent lines.

25. Given a plane and two points on the same side of it. Find the point in the plane the sum of the distances of which from the given points is a minimum. (Similar to Problem 1.16.)

26. If a straight line is equally inclined to each of three straight lines in a plane, it is perpendicular to the plane in which they lie.

27. Draw two parallel planes, one through each of two straight lines which do not meet and are not parallel. (Let $AB$, $CD$ be the lines. Draw $AE \parallel CD$, $CF \parallel AB$. Hence plane $AEB$ is $\parallel$ plane $CFD$.)

28. If two straight lines are parallel, each is parallel to every plane passing through the other.

29. Draw a line which will be parallel to one given line and intersect two other given lines, no two of the given lines meeting.
30. In a tetrahedron in which each edge is equal to the opposite edge (called an *isosceles tetrahedron*) prove that (a) the faces are congruent; (b) the lines joining the mid-points of opposite edges are perpendicular to the edges which they bisect and to one another. [(a) Each face has sides $a, b, c$. (b) Let $L, L', M, M', N, N'$ bisect $AB, CD, AC, BD, AD, BC$ of tetrahedron $ABCD$. Then $LM = \frac{1}{2} BC = M'L'$ and $LM' = \frac{1}{2} AD = ML'$. Hence $LML'M'$ is a rhombus. Hence $LL', MM'$ are perpendicular, so $MM', NN'$ and $NN', LL'$. Now $NN'$ is perpendicular to $LL'$ and $MM'$ and therefore to $LM$ and $L'M$, i.e., to $BC$ and $AD$.]

31. Prove that a sphere can be inscribed in an isosceles tetrahedron and that the lines joining its corners with the centroids of the faces meet at the center of this sphere.

32. The six planes bisecting the edges of a tetrahedron perpendicularly meet in a point, which is the center of the circumscribing sphere.

33. If a pair of points is taken on each edge of a tetrahedron such that the pairs on any two adjacent edges are concyclic, show that the twelve points lie on the same sphere.

34. The section of a tetrahedron by a plane is a parallelogram if and only if the plane is parallel to a pair of opposite edges. (Let the parallelogram be $LML'M'$, where $L, L', M, M'$ are on $AB, CD, AC, BD$ of a tetrahedron $ABCD$. Then the three planes $LML'$, $ABC$, $DBC$ meet at a point; and this is at infinity since $LM, M'L'$ are $\parallel$. Hence $LM$ is $\parallel BC$ and so $LM'$ to $AD$.)

35. In a tetrahedron the lines (called the *medians*) joining the mid-points of opposite edges bisect one another at the centroid.

36. In Theorem 8.28 of a tetrahedron $ABCD$, the volumes subtended by the faces at $G$ are equal, $G$ being the centroid of the tetrahedron. (For $ABCD : GBCD = AA' : GA' = 4 : 1$, and so on.)

37. Show that the shortest distance between two opposite edges of a regular tetrahedron is equal to half the diagonal of the square described on the edge.

38. Two planes which are not parallel are cut by two parallel planes. Prove that the lines of section of the first two with the last two contain equal angles.

39. If two parallel planes are cut by three other planes which have no line common to all three, and no two of which are parallel, the triangles formed by the intersections of the parallel planes with the three other planes are similar to each other.

40. Two straight lines do not intersect and are not parallel. Find a plane upon which their projections will be parallel. (Refer to Exercise 27.)

41. Two similar polygons, not in the same plane, are placed with their homologous sides parallel. Prove that the straight lines which join corresponding vertices are either parallel or concurrent.
42. Show that a tetrahedron can be formed of any four equal congruent triangles, provided the triangles are acute-angled.

43. Of parallelograms which are parallel to two opposite edges of a tetrahedron, the one of greatest area bisects each edge. (For in a tetrahedron $ABCD$; $AL : AB = LM : BC$, i.e., $LM \propto AL$. So $LM' \propto BL$. See Exercise 34 for the parallelogram $LML'M'$ on $AB$, $CD$, $AC$, $BD$. Also the angle $MLM'$ is constant, being the angle between $BC$ and $AD$. Hence the area varies as $AL \cdot LB$.)

44. If a solid angle at $O$ is contained by three plane angles $AOB$, $BOC$, $COA$ and $D$ is any point in the plane $ABC$, prove that the angles $AOB$, $BOC$, $COA$ are together less than twice the angles $AOD$, $BOD$, $COD$.

45. Divide a straight line similarly to a given divided straight line lying in a different plane.

46. Two straight lines which intersect are inclined to each other at an angle equal to $\frac{\pi}{4}$ of a right angle and to a given plane, each at an angle equal to half a right angle. Prove that their projections on this plane are at right angles to each other.

47. If perpendiculars to two faces of a tetrahedron from the opposite vertices intersect, prove that the edge in which the faces intersect is perpendicular to the opposite edge. (Refer to Problem 8.13.)

48. A pyramid is described with a parallelogram as base. Show that if its four triangular faces are equal in area, their projections on the base are the triangles into which the parallelogram is divided by its diagonals; and the parallelogram is a rhombus.

49. Through one of the diagonals of a parallelogram a plane is drawn. Prove that the perpendiculars let fall from the end of the other diagonal on this plane are equal.

50. Prove that the greatest tetrahedron which can be inscribed in a given sphere is equilateral. (Take $B, C, D$ fixed in a tetrahedron $ABCD$. Then the perpendicular from $A$ on $BCD$ must be greatest; i.e., $A$ must be on the diameter of the sphere through the circumcenter of $BCD$. Hence $AB = AC = AD$.)

51. Prove that a sphere can be described through two circles in different planes, provided these circles have two common points (i.e., through the common points and a point on each circle).

52. Prove that a circle $PQR$ and its inverse with respect to a point, not in its plane, lie on the same sphere (i.e., the sphere $PQRP'$).

53. $A$ and $B$ are two given points on two given planes which intersect in the line $l$. Find a point $P$ on $l$ such that $AP + PB$ may be least.

54. The points $X$ and $Y$ move on given lines which are not in the same plane. Show that the center of $XY$ moves on a plane.

55. Draw a plane to cut the given lines $OA, OB, OC, OD$, no three of which are in the same plane, in the parallelogram $ABCD$. 


56. Prove that in the tetrahedron of Problem 8.13, each altitude meets the opposite face at its orthocenter.

57. Prove that in the tetrahedron of Problem 8.13, a sphere can be drawn through the centers of its edges and the feet of the shortest distances between opposite edges.

58. If a plane cuts a sphere and the area of the circle of intersection is \( \frac{1}{16} \) of the surface area of the sphere, find the distance of this plane from the center of the sphere in terms of its radius.

59. Prove that in an isosceles tetrahedron, where every two opposite edges are equal, the three angles at any vertex are together equal to two right angles.

60. In an isosceles tetrahedron, prove that the centers of the inscribed and escribed spheres coincide. (Refer to Exercise 31.)

61. In an isosceles tetrahedron, show that each plane angle of every face is acute.

62. In an equilateral tetrahedron \( ABCD \), \( E \) bisects the perpendicular from \( A \) on \( BCD \). Show that \( EB, EC, ED \) are mutually perpendicular. [Let \( AG \) be the \( \perp \) from \( A \) on \( BCD \). Then, by symmetry, \( G \) is the centroid of the equilateral \( \triangle BCD \) and \( AG \perp BG \). \( \therefore 4BE^2 = 4BG^2 + 4GE^2 \). But, \( BG = \frac{1}{3} BL \) (if \( BL \perp CD \)) = \( \frac{1}{3} BC(\sqrt{3}/2) \). \( \therefore BG = BC/\sqrt{3} \) and, \( 4GE^2 = AG^2 = AB^2 - BG^2 \). \( \therefore 4BE^2 = (4BC^2/3) + AB^2 - (BC^2/3) = AB^2 + BC^2 = 2BC^2 = 4CE^2 = 4DE^2 \). Hence \( BE^2 + CE^2 = 2BE^2 = BC^2 \). Hence \( BE \perp CE \), so \( CE \perp DE \) and \( DE \perp BE \).]

63. Show that all parallel sections of an isosceles tetrahedron which are parallelograms have the same perimeter.

64. A plane is drawn through an edge of a tetrahedron and the center of the opposite edge. Show that the six planes so drawn meet in a point.

65. Prove that the sum of the squares of the edges of a tetrahedron is equal to four times the sum of the squares of the lines joining the mid-points of opposite edges.

66. If the lines joining corresponding vertices of two tetrahedrons meet in a point, the intersections of corresponding edges lie on a plane.

67. If a sphere can be drawn to touch the six edges of a tetrahedron, the sum of each two opposite edges is the same.

68. If the section of a tetrahedron by a plane is a square, show that two opposite edges must be perpendicular and, in that case, show how to construct the square. (With the figure of Exercise 34, \( LM \parallel BC \) and \( LM' \parallel AD \). Hence, if \( LML'M' \) is a square so that \( LM \perp LM' \), then \( BC \perp AD \). Again, if \( LML'M' \) is a square \( LM = LM' \). But, \( LM : BC = AL : AB \) and \( LM' : AD = BL : AB \). Hence \( BC \cdot AL = AD \cdot BL \). Hence to get \( L \) we divide \( AB \) so that \( AL : BL = AD : BC \). Then, draw \( LM \parallel BC, ML' \parallel AD, \) and \( L'M' \parallel BC \).)
69. Prove that the three lines joining the middle points of opposite edges of a tetrahedron are concurrent and that they form, two and two, the diagonals of three parallelograms, such that the angles between their sides are the angles between the opposite edges of the tetrahedron.

70. A pyramid stands on a quadrilateral base. Draw a plane such that its lines of intersection with the faces of the pyramid may form a parallelogram in which one angle is given.

71. Prove that the sections of a tetrahedron made by planes parallel to a pair of opposite edges are rectangles if the opposite edges are at right angles.

72. Prove that if the line joining one vertex of a tetrahedron to the orthocenter of the opposite face be perpendicular to that face, the same is true with regard to the other vertices.

73. Three planes intersect in a point, and a plane is drawn through the common section of each pair perpendicular to the third plane. Show that these three planes intersect in a straight line.

74. If through the edges $AB$, $AC$, $AD$ of a tetrahedron $ABCD$ planes are drawn such that their intersections with the planes $CAD$, $DAB$, $BAC$ respectively bisect the angles at $A$ in those planes, these planes have a common line of intersection.

75. If $BAC$, $CAD$, $DAB$ are three plane angles containing a solid angle, prove that the angle between $AD$ and the straight line bisecting the angle $BAC$ is less than half the sum of the angles $BAD$, $CAD$.

76. A solid angle is contained by three plane angles $BOC$, $COA$, $AOB$. If $BOC$, $COA$ are together equal to two right angles, prove that $CO$ is perpendicular to the line which bisects the angle $AOB$.

77. $ABCD$ is a tetrahedron; $AD$, $BD$, $CD$ are cut in $L$, $M$, $N$ by a plane parallel to $ABC$. $BN$, $CM$ meet in $P$; $CL$, $AN$ meet in $Q$; $AM$, $BL$ meet in $R$. Show that the triangles $LMN$, $PQR$ are similar.

78. $ABCD$ is a face and $AE$ a diagonal of a cube. $BG$ is drawn perpendicular to $AE$ and $DG$ is joined. Prove that $DG$ is perpendicular to $AE$.

79. Given three straight lines not in the same plane. Draw through a given point a straight line equally inclined to the three. (Through given point $O$ draw $OA$, $OB$, $OC \parallel$ to the given skew lines. Bisect $\angle s$ $AOB$, $BOC$ by $OD$, $OE$; through $OD$, $OE$ draw planes $ODF$, $OEF \perp$ planes $AOB$, $BOC$. Hence any line through $O$ in plane $ODF$ makes equal angles with $OA$, $OB$ and any line through $O$ in plane $OEF$ makes equal angles with $OB$, $OC$. Hence $OF$ is equally inclined to given lines.)

80. On the same side of the plane of a parallelogram $ABCD$, three straight lines $Bb$, $Cc$, $Dd$ are drawn in parallel directions, and such that $Cz$ is equal to $Bb$ and $Dd$ together. Prove that the points $A$, $b$, $c$, $d$ lie in one plane.
81. In the tetrahedron \( OABC \), \( OA \), \( OB \), \( OC \) are equal. Prove that each of them is greater than the radius of the circle circumscribing the triangle \( ABC \).

82. Prove that in any tetrahedron, the plane bisecting any dihedral angle divides the opposite edge into segments which are in the ratio of the areas of the adjacent faces. If each edge of a tetrahedron is equal to the opposite edge, the plane bisecting any dihedral angle bisects the opposite edge.

83. A pyramid stands on a triangular base. Show that if the circles inscribed in three of the faces touch each other, the circle inscribed in the fourth face will touch all the others.

84. If \( ABCD \) be a tetrahedron and \( G \) the intersection of lines drawn from \( A, B, C \) to the middle points of \( BC, CA, AB \) respectively, prove that the squares on \( AD, BD, CD \) are together equal to the squares on \( AG, BG, CG \) with three times the square on \( DG \).

85. In a tetrahedron the straight line joining the middle points of one pair of opposite edges is at right angles to both edges. Prove that of the other four edges any one is equal in length to the opposite one.

86. \( AB, AF, CB, CD \) are edges of a cube, \( AF, CD \) being opposite edges. Find the angles between the planes \( ABD, AFD, ABF \). (Let \( ABEF \) be a face of the cube. \( BD, BE \) are both \( \perp AB \). \( \therefore \) \( EBD \) is the angle between planes \( ABD, ABF \) and \( \angle EBD = \frac{1}{2} \) right angle. Similarly, the angle between planes \( AFD, ABF = \frac{1}{4} \) right angle. Draw \( BG \perp AD \). \( \therefore DA \cdot AG = AB^2 = AF^2 \). \( \therefore FG \) is \( \perp AD \) and \( BG = GF \). Draw \( GH \perp BF \). \( \therefore BH = HF \). \( \therefore H \) is the intersection of \( AE, BF \). Again, \( GH : AH = DE : AD = AB : AD = BG : BD \) and \( BD = AE = 2 \) AH. \( \therefore BG = 2 \) GH. \( \therefore \angle BGH = \frac{2}{3} \) right angle. \( \therefore \angle BGF = \frac{4}{3} \) right angle.)

87. If \( P \) be a point equally distant from the vertices \( A, B, C \) of a right-angled triangle, of which \( A \) is the right angle and \( D \) the middle point of \( BC \), prove that \( PD \) is at right angles to the plane \( ABC \). Prove also that the angle between the planes \( PAC, PBC \) and the angle between the planes \( PAB, PBC \) are together equal to the angle between the planes \( PAC, PAB \).

88. \( ABCD \) is a tetrahedron, \( EFGH \) a plane section cutting \( AC \) in \( F \). Prove that if the section is a rhombus in a plane parallel to \( BC \) and \( AD \), then \( BC : AD = CF : AF \).

89. Under what circumstances is it possible to draw a plane so as to cut the faces of a tetrahedron in four lines forming a square?

90. Prove that if each solid angle at the base of a tetrahedron is contained by three plane angles which make up two right angles, the tetrahedron will have all its faces equal and similar.

91. Prove that in an isosceles tetrahedron where every pair of opposite edges are equal, the inscribed and circumscribed spheres are concentric.
92. If a cone in which the generator equals the diameter of the base touches a sphere along the contour of the circle of the base, and a cylinder envelops the sphere and the cone so that the vertex of the cone lies in the upper face of cylinder, find the ratio of the total surface areas and the ratio of the volumes of these three sides.

93. $O_1, O_2, O_3, O_4$ are the centers of the circles inscribed in the faces $BCD$, $ACD$, $ABD$, $ABC$ respectively of the tetrahedron $ABCD$. Prove that $AO_1$, $BO_2$, $CO_3$, $DO_4$ will meet in a point if the rectangles $AB \cdot CD$, $AC \cdot BD$ and $AD \cdot BC$ are all equal.

94. Prove that a sphere can be described through two circles which touch one another but are not in the same plane. (This is the limit of Exercise 51, when the two points coincide.)

95. Prove that the cone joining any point to a circular section of a sphere cuts the sphere again in a circle.

96. Prove that four planes can be drawn through the center of a cube, each of which cuts six edges of the cube in the corners of a regular hexagon and the other six edges produced in the corners of another regular hexagon. The area of the second hexagon is three times that of the first and the sides of the second are perpendicular to the central radii drawn to the corners of the first.

97. A cone is described such that its base circle touches internally the sides of a square base of a regular pyramid which has the same vertex as the cone. If the common altitude of cone and pyramid is $\sqrt{3}/2$, the side of the square base, find the ratio of the volume of the sphere inscribed in the cone to the volume of solid between pyramid and cone.

98. Prove that every two circular sections of a cone are either parallel or inverse with respect to the vertex.

99. Find the locus of a point in a given plane whose distances from the given points, which are not in the plane, are in a given ratio. (Let $A, B$ be the given points and $P$ the variable point. Then since $PA : PB$ is given, the locus of $P$ is a sphere. The required locus is the circle which is the section of this sphere by the given plane.)

100. Draw a plane to touch three given spheres. (Let $S$ be a center of similitude of the spheres 1, 2, and $S'$ of 1, 3. Then a tangent plane through the line $SS'$ to 1 will touch 2, 3.)
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